Simply typed and untyped Lambda Calculus revisited

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Abstract

In [6] one finds a general method to describe various (typed) λ-calculi categorically. Here we give an elementary formulation in terms of indexed categories of the outcome of applying this method to the simply typed λ-calculus. It yields a categorical structure in which one can describe exponent types without assuming cartesian product types. Specializing to the "monoid" case where one has only one type yields a categorical description of the untyped λ-calculus.

In the literature there are two categorical notions for the untyped λ-calculus: one by Obtulowicz and one by Scott & Koymans. The notion we arrive at subsumes both of these; it can be seen as a mild generalization of the first one.

1. Introduction

The straightforward way to describe the simply typed λ-calculus (denoted here by $\lambda 1$) categorically is in terms of cartesian closed categories (CCC's), see [10]. On the type theoretic side this caused some discomfort, because one commonly uses only exponent types without assuming cartesian product types — let alone unit (*i.e.* terminal) types. The typical reply from category theory is that one needs cartesian products in order to define exponents. Below we give a categorical description of exponent types without assuming cartesian product types. We do use cartesian products of contexts; these always exist by concatenation. Thus both sides can be satisfied by carefully distinguishing between cartesian products of types and cartesian products of contexts. We introduce an appropriate categorical structure for doing so.

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In [6] one can find a general method to describe typed λ-calculi categorically. One of the main organizing principles used there is formulated below. It deserves the status of a slogan.

> A context is an index for the category of types and terms derivable in that context.

This work can be read as the result of applying the slogan to the simply typed and untyped λ-calculus. It naturally leads us to the definition of a $\lambda 1$-category in 3.1. In [6] these matters can be found in terms of so-called "constant comprehension categories". Here we reformulate them in terms of the more elementary indexed categories. We like to mention a few advantages of this approach.

- It is conceptually satisfying in the sense that it follows the above slogan.
- It allows a description of exponent types without assuming cartesian product types. The exponent types are described by *right* adjoints and the cartesian product types by *left* adjoints to so-called weakening functors (see lemmas 3.3 and 3.7).
- Non-extensional versions are obtained by simply replacing the above adjoints by semi-adjoints. This follows a general categorical understanding of non-extensionality, see [5].
- The untyped λ-calculus is covered by the "monoid" subcase where one has only one type. This follows Dana Scott's insistance that the untyped λ-calculus should be considered as a special case of the typed λ-calculus, *viz.* as a calculus with a single type. Again this is in line with a general categorical understanding of untyped as described by the monoid case.

The force of this last point is that in our approach the untyped λ-calculus can be treated in a corollary. The outcome appears to be (by lemma 4.4) essentially the same notion as introduced by Obtulowicz, see [11, 12]. There is a slight difference though, which concerns the fact that where we use an arbitrary cartesian category of contexts, Obtulowicz restricts himself to an algebraic theory. In the fourth section on the untyped λ-calculus one finds theorem 4.10 which yields an adjunction between set-theoretical λ-algebras (as described in [1]) and our categorical λ-algebras. The main ingredients of this theorem already occur in [2, constructions V and VI in 1.5]. Hence with respect to this fourth section, we only have a very moderate claim to originality. The best way to see it is probably as providing an *a posteriori* justification for the early definition of Obtulowicz.

It is worthwhile noticing that the abovementioned theorem 4.10 could not be obtained in the Scott-Koymans approach to the untyped λ-calculus (see example 4.8 (iii)): turning a CCC with a reflexive object first into a λ-algebra and then again into a category yields incomparable results. This is due to the use of the Karoubi-envelope: it introduces too many new objects and arrows, see [9, 1]. In the approach described here, this Karoubi-construction can be avoided altogether.

Finally we mention some general points. Since the categories we use are close to syntax we describe categorical structure "on-the-nose", *i.e.* up-to-equality and not up-to-isomorphism. For example, **CCC** denotes the category of categories with distinguished terminals, cartesian products and exponents; the morphisms in this category are functors which preserve this structure on-the-nose. Similarly an indexed category is really a functor (and not just a pseudo functor). Further, aspects of size will be ignored. In an adjoint situation the transpose operations will often be denoted by $(-)^{\vee}$ and $(-)^{\wedge}$.

2. Preliminaries

Let C be a non-empty set of "constants". The simply typed λ-calculus $\lambda 1(C)$ has as set of types the smallest collection $Type_C$ satisfying $C \subseteq Type_C$ and $\sigma, \tau \in Type_C \;\Rightarrow\; (\sigma \to \tau) \in Type_C$. Hence there are no type variables and no cartesian product types. Terms are then formed by the usual context projection rule and the rules for application and abstraction:

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{MN : \tau} \qquad\qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma.\, M : \sigma \to \tau}$$

We mention the (β)- and (η)-conversions

$$(\lambda x : \sigma.\, M)N \;=\; M[x := N] \qquad\qquad \lambda y : \sigma.\, My \;=\; M,$$

where in the latter case the variable y is supposed not to be free in M.

An *indexed category* is a functor of the form $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$. The category $\mathbf{B}$ is then called the *base*; a category ΨA (for $A \in \mathbf{B}$) is called a *fibre* (above A). For every morphism $u : A \to B$ in $\mathbf{B}$ one has a so-called *reindexing* or *subsitution* functor $\Psi u : \Psi B \to \Psi A$; notice the reverse direction. If no confusion arises one often writes u^* for Ψu. An elementary example is given by the functor $I \mapsto \mathbf{C}^I$, which maps a set I to the category of I-indexed collections of objects and arrows of an arbitrary category $\mathbf{C}$.
A *morphism of indexed categories* from $\Phi : \mathbf{A}^{op} \to \mathbf{Cat}$ to $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$

consists of a pair (H, α), where H is a functor $\mathbf{A} \to \mathbf{B}$ and α is a natural transformation $\Phi \stackrel{\cdot}{\longrightarrow} \Psi H^{op}$. In this way one obtains a category $\mathbf{IC}$.

The categorical basis for the whole of this work is what we like to call a *CT-structure* — where 'C' stands for 'context' and 'T' for 'type'. It is given by an arbitrary cartesian category $\mathbf{B}$ of "contexts" (*i.e.* a category with distinguished finite products) together with a collection of "types"

$$T \;\subseteq\; Obj\,\mathbf{B}.$$

The inclusion of types into contexts can be understood as an identification of types with the corresponding singleton contexts. A *morphism of CT-structures* from $(\mathbf{A}, S)$ to $(\mathbf{B}, T)$ consists of a functor $H : \mathbf{A} \to \mathbf{B}$ which preserves the distinguished finite products on-the-nose and maps types to types, *i.e.* satisfies $H[S] \subseteq T$. The resulting category will be denoted by **CT**.
We mention some basic matters concerning CT-structures.

(1) A CT-structure $(\mathbf{B}, T)$ will be called *non-trivial* if there is at least one $X \in T$ such that the hom-set $\mathbf{B}(t, X)$ is non-empty — where t is the terminal in $\mathbf{B}$. This gives a non-degeneracy condition.

(2) A *monoid* CT-structure is one of the form $(\mathbf{A}, \{\Omega\})$. It describes a type-free setting.

(3) For a CT-structure $(\mathbf{B}, T)$, let $\overline{T} \subseteq Obj\,\mathbf{B}$ be the smallest collection satisfying

$$\begin{array}{rcl} T & \subseteq & \overline{T} \\ t & \in & \overline{T} \\ A, B \in \overline{T} & \Rightarrow & (A \times B) \in \overline{T} \end{array}$$

i.e. $\overline{T}$ is obtained from T by adding finite products. One can then form the full subcategory of $\mathbf{B}$ determined by $\overline{T}$; it will also be denoted by $\overline{T}$. A CT-structure $(\mathbf{B}, T)$ will now be called *full* if there is an equivalence of categories $\overline{T} \simeq \mathbf{B}$. Then every context is a finite product of types.

(4) For a CT-structure $(\mathbf{B}, T)$, one can define a so-called *constant* indexed category of the form $\mathcal{K}_T : \mathbf{B}^{op} \to \mathbf{Cat}$ as follows. For $A \in \mathbf{B}$ the fibre category $\mathcal{K}_T(A)$ has elements $X \in T$ as objects. A morphism $f : X \to Y$ in $\mathcal{K}_T(A)$ is a map $f : A \times X \to Y$ in $\mathbf{B}$. Composition of $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{K}_T(A)$ — *i.e.* of $f : A \times X \to Y$ and $g : A \times Y \to Z$ in $\mathbf{B}$ — is given by $g \bullet f = g \circ \langle \pi, f \rangle : A \times X \to A \times Y \to Z$. The identity map on X is $\pi' : A \times X \to X$. Thus one obtains a category. For $u : B \to A$ in $\mathbf{B}$ there is a reindexing functor $u^* = \mathcal{K}_T(u) : \mathcal{K}_T(A) \to \mathcal{K}_T(B)$ given by $X \mapsto X$ and $f \mapsto f \circ u \times id$.

This indexed category $\mathcal{K}_T : \mathbf{B}^{op} \to \mathbf{Cat}$ is called "constant" because the objects in the fibres $\mathcal{K}_T(A)$ do not depend on the index A.
Recall that $T \subseteq Obj\,\mathbf{B}$ is a parameter in this construction. In two extreme cases, we adapt the notation a bit: if $T = Obj\,\mathbf{B}$ we write $\mathcal{K}_{\mathbf{B}}$ for $\mathcal{K}_{Obj\mathbf{B}}$ and if T is a singleton, say $T = \{\Omega\}$, we write $\mathcal{K}_\Omega$ for $\mathcal{K}_{\{\Omega\}}$.
This construction extends to a functor $\mathbf{CT} \to \mathbf{IC}$: for a morphism $H : (\mathbf{A}, S) \to (\mathbf{B}, T)$ of CT-structures one obtains a morphism $(H, \widetilde{H}) : \mathcal{K}_S \to \mathcal{K}_T$. The natural transformation $\widetilde{H} : \mathcal{K}_S \stackrel{\cdot}{\longrightarrow} \mathcal{K}_T H^{op}$ has component functors $(\widetilde{H})_A : \mathcal{K}_S(A) \to \mathcal{K}_T(HA)$ given by $X \mapsto HX$ and $f \mapsto Hf$.

(5) There is an elementary form of quantification for indexed categories associated with a CT-structure $(\mathbf{B}, T)$. We say that an indexed category $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ has *T-products* (*resp.* *T-sums*) if both

- for each $A \in \mathbf{B}$ and $X \in T$, the reindexing functor
$$\pi^*_{A,X} \;:\; \Psi(A) \;\;\to\;\; \Psi(A \times X)$$
has a right adjoint $\Pi_{(A,X)}$ (*resp.* a left adjoint $\Sigma_{(A,X)}$);

- the "Beck-Chevalley" condition holds, *i.e.* for each $u : B \to A$ in $\mathbf{B}$ and $X \in T$, the pair $(u^*, (u \times id)^*)$ is a morphism of adjunctions from $\pi^*_{A,X} \dashv \Pi_{(A,X)}$ to $\pi^*_{B,X} \dashv \Pi_{(B,X)}$ in
$$\begin{tikzcd}
\Psi(A) \arrow[rr, "u^*"] \arrow[d, shift right=2, "\pi^*_{A,X}"'] \arrow[d, phantom, "\dashv"] & & \Psi(B) \arrow[d, shift right=2, "\pi^*_{B,X}"'] \arrow[d, phantom, "\dashv"] \\
\Psi(A \times X) \arrow[rr, "(u \times id)^*"'] \arrow[u, shift right=2, "\Pi_{(A,X)}"'] & & \Psi(B \times X) \arrow[u, shift right=2, "\Pi_{(B,X)}"']
\end{tikzcd}$$
(*resp.* the pair $((u \times id)^*, u^*)$ is a morphism of adjunctions from $\Sigma_{(A,X)} \dashv \pi^*_{A,X}$ to $\Sigma_{(B,X)} \dashv \pi^*_{B,X}$).

The corresponding notion of morphism is as follows. Let $\Phi : \mathbf{A}^{op} \to \mathbf{Cat}$ have S-products (*resp.* sums) and $\Psi : \mathbf{B}^{op} \to \mathbf{Cat}$ have T-products (*resp.* sums). A morphism (H, α) of indexed categories from Φ to Ψ is said to *preserve* products (*resp.* sums) if

- $H : (\mathbf{A}, S) \to (\mathbf{B}, T)$ is a morphism of CT-structures;

- for every $A \in \mathbf{A}$ and $X \in S$, the pair $\langle \alpha_{A\times X} : \Phi(A \times X) \to \Psi(HA \times HX),\ \alpha_A : \Phi(A) \to \Psi(HA)\rangle$ is a map of adjunctions from $\pi^*_{A,X} \dashv \Pi_{(A,X)}$ to $\pi^*_{HA,HX} \dashv \Pi_{(HA,HX)}$
(*resp.* $\langle \alpha_A, \alpha_{A\times X}\rangle$ from $\Sigma_{(A,X)} \dashv \pi^*_{A,X}$ to $\Sigma_{HA,HX} \dashv \pi^*_{HA,HX}$).

(6) In order to handle non-extensionality one uses the "semi-adjunctions" from [5]. To obtain suitable notions of semi-product/sum and of morphisms of these, one simply replaces in (5) adjunction by semi-adjunction and map of adjunction by map of semi-adjunction.
These non-extensional concepts will only be used in the fourth section on the untyped λ-calculus.

Convention. In case the above collection T consists of a singleton, say $T = \{\Omega\}$, we speak about Ω-products instead of $\{\Omega\}$-products.
We often loosely say that a constant indexed category $\mathcal{K}_T : \mathbf{B}^{op} \to \mathbf{Cat}$ has products/sums when we mean T-products/sums.

One standardly calls right adjoints to the "weakening" functors π^* *products*; left adjoints are called *sums*. In the constant case (without type dependency) the Π becomes $\to$ and the Σ becomes $\times$. This will be shown in the next section. The Beck-Chevalley requirement guarantees the proper distribution of substitution over Π's and Σ's and over the associated operations.

3. Simply typed λ-calculus

We start with a categorical structure which is appropriate for the simply typed λ-calculus. It will be investigated in the rest of this section.

Definition 3.1 (i) *A* λ1-category *is a non-trivial CT-structure* $(\mathbf{B}, T)$ *such that the corresponding constant indexed category* $\mathcal{K}_T : \mathbf{B}^{op} \to \mathbf{Cat}$ *has products.*
(ii) *A* morphism of λ1-categories *from* $(\mathbf{A}, S)$ *to* $(\mathbf{B}, T)$ *is given by a morphism* $H : (\mathbf{A}, S) \to (\mathbf{B}, T)$ *of CT-structures such that the corresponding morphism of indexed categories* $(H, \widetilde{H}) : \mathcal{K}_S \to \mathcal{K}_T$ *preserves products.*

Before submitting the notion of a λ1-category to a close analysis, we present a term model example (*i.e.* a generalized Lindenbaum - Tarski construction); it is particularly suitable to convey the type theoretic intuition.

Example 3.2 Let's consider the calculus $\lambda 1(C)$ for a given non-empty set of constants C. A base category $\mathbf{C}$ is formed with contexts $\Gamma = x_1 : \sigma_1 \ldots x_n : \sigma_n$ as objects. A morphism $\Delta \to \Gamma$ in $\mathbf{C}$ — with Γ as before — consists of an n-tuple of $\beta\eta$-equivalence classes of terms $\langle [M_1], \ldots, [M_n]\rangle$ where $\Delta \vdash M_i : \sigma_i$. Composition is done by substitution: if $\langle [N_1], \ldots, [N_m]\rangle : \Gamma \to \Theta$, then $\langle [N_1], \ldots, [N_m]\rangle \circ \langle [M_1], \ldots, [M_n]\rangle = \langle [L_1], \ldots, [L_m]\rangle$ with $L_i = N_i[\vec{x} := \vec{M}]$. The identity on Γ is $\langle [x_1], \ldots, [x_n]\rangle$.
The category $\mathbf{C}$ has finite products: the empty context is terminal and concatenation of contexts yields cartesian products.

The collection of types in $\lambda 1(C)$ is denoted here by $T = \mathit{Type}_C$. By identifying a type σ with the context $x : \sigma$, one has $T \subseteq \mathit{Obj}\,\mathbf{C}$. Notice that the resulting CT-structure $(\mathbf{C}, T)$ is non-trivial because the set of constants is non-empty. As a result we have an indexed category $\mathcal{K}_T : \mathbf{C}^{op} \to \mathbf{Cat}$. The fibre $\mathcal{K}_T(\Gamma)$ above context Γ has types and terms in context Γ as objects and morphisms: objects are types $\sigma \in T$ and morphisms $\sigma \to \tau$ in $\mathcal{K}_T(\Gamma)$ are terms $[P(y)]$ with $\Gamma, y{:}\,\sigma \vdash P{:}\,\tau$. Substitution functors between the fibres are obtained by substitution: if $\langle [M_1], \ldots, [M_n] \rangle : \Delta \to \Gamma$ in $\mathbf{C}$ as above, then $\langle [M_1], \ldots, [M_n] \rangle^* : \mathcal{K}_T(\Gamma) \to \mathcal{K}_T(\Delta)$ is defined by $\sigma \mapsto \sigma$ and $[P] \mapsto [\, P[\vec{x} := \vec{M}]\,]$. All this follows from the construction of constant indexed categories described in (4) in the preliminary section.
Finally we have to show that $\mathcal{K}_T : \mathbf{C}^{op} \to \mathbf{Cat}$ has products. For a context Γ and a type σ we have to define a functor $\Pi_{(\Gamma,\sigma)} : \mathcal{K}_T(\Gamma, y{:}\,\sigma) \to \mathcal{K}_T(\Gamma)$. This is done on objects by $\tau \mapsto \sigma \to \tau$. By abstraction and application one easily obtains an isomorphism

$$\mathcal{K}_T(\Gamma, y{:}\,\sigma)\big(\pi^*_{\Gamma,y:\sigma}(\rho),\ \ \tau\big) \quad \cong \quad \mathcal{K}_T(\Gamma)\big(\rho,\ \ \Pi_{(\Gamma,\sigma)}(\tau)\big).$$

Remaining verifications (including the Beck-Chevalley condition) are left to the reader.

The next result shows that products for a constant indexed category can be described simply in terms of exponents. The result (a) $\Rightarrow$ (b) follows from lemma 5.4 in [8]. The formalism used there is somewhat different; therefore we give the details here.

Lemma 3.3 *Let* $(\mathbf{B}, T)$ *be a non-trivial CT-structure. The following two statements are equivalent.*
(a) $(\mathbf{B}, T)$ *is a* $\lambda 1$*-category.*
(b) *The collection* T *is closed under exponents in the following sense. For "types"* $X, Y \in T$ *there is an exponent type* $X \Rightarrow Y \in T$ *together with a map* $ev : (X \Rightarrow Y) \times X \to Y$ *such that for each* $A \in \mathbf{B}$ *and* $f : A \times X \to Y$ *in* $\mathbf{B}$ *there is a unique map* $\Lambda(f) : A \to X \Rightarrow Y$ *with* $ev \circ \Lambda(f) \times id = f$.

Proof : (b) $\Rightarrow$ (a) For $A \in \mathbf{B}$ and $X \in T$ one defines $\Pi_{(A,X)} : \mathcal{K}_{\mathbf{B}}(A \times X) \to \mathcal{K}_{\mathbf{B}}(A)$ by $Y \mapsto X \Rightarrow Y$. Then indeed,

$$\begin{aligned}
\mathcal{K}_T(A \times X)\big(\,\pi^*_{A,X}(Z),\ \ Y\,\big) \quad &= \quad \mathcal{K}_T(A \times X)\big(\,Z,\ \ Y\,\big) \\
&= \quad \mathbf{B}\big(\,(A \times X) \times Z,\ \ Y\,\big) \\
&\cong \quad \mathbf{B}\big(\,(A \times Z) \times X,\ \ Y\,\big) \\
&\cong \quad \mathbf{B}\big(\,A \times Z,\ \ X \Rightarrow Y\,\big) \\
&= \quad \mathcal{K}_T(A)\big(\,Z,\ \ X \Rightarrow Y\,\big) \\
&= \quad \mathcal{K}_T(A)\big(\,Z,\ \ \Pi_{(A,X)}(Y)\,\big).
\end{aligned}$$

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Hence one has an adjunction $\pi^*_{A,X} \dashv \Pi_{(A,X)}$.

(a) $\Rightarrow$ (b) For types $X, Y \in T$ one has $Y \in \mathcal{K}_T(t \times X)$ and thus we define

$$X \Rightarrow Y \quad = \quad \Pi_{(t,X)}(Y) \quad \in \quad T.$$

We notice that for an object $A \in \mathbf{B}$ one has $!_A : A \to t$ in $\mathbf{B}$ and thus

$$\begin{aligned}
X \Rightarrow Y \quad = \quad !^*_A(X \Rightarrow Y) \quad &= \quad !^*_A\big(\Pi_{(t,X)}(Y)\big) \\
&= \quad \Pi_{(A,X)}((!_A \times id)^*(Y)) \qquad \text{by Beck-Chevalley} \\
&= \quad \Pi_{(A,X)}(Y).
\end{aligned}$$

The unit and counit of the adjunction $\pi^*_{A,X} \dashv \Pi_{(A,X)}$ are maps $\eta^{(A,X)}_Z : A \times Z \to \Pi_{(A,X)}(Z)$ and $\varepsilon^{(A,X)}_Y : (A \times X) \times \Pi_{(A,X)}(Y) \to Y$ in $\mathbf{B}$. Hence in $\mathcal{K}_T(t \times X)$ one has $\varepsilon^{(t,X)}_Y : (t \times X) \times \Pi_{(t,X)}(Y) \to Y$. We define

$$ev_{X,Y} \quad = \quad \varepsilon^{(t,X)}_Y \quad \circ \quad \langle\langle !,\ \pi' \rangle,\ \pi \rangle \quad : \quad (X \Rightarrow Y) \times X \ \to \ Y.$$

The definition of abstraction is a bit tricky. Since $(\mathbf{B}, T)$ is by assumption non-trivial, we may assume an object $Z \in T$ with an arrow $z_0 : t \to Z$. For a map $f : A \times X \to Y$ in $\mathbf{B}$ (with $A \in \mathbf{B}$ and $X, Y \in T$) one has $f \circ \pi : (A \times X) \times Z \to Y$ in $\mathbf{B}$ and thus $f \circ \pi : \pi^*_{A,X}(Z) \to Y$ in $\mathcal{K}_T(A \times X)$. Taking the transpose accross the adjunction $\pi^*_{A,X} \dashv \Pi_{(A,X)}$ yields $(f \circ \pi)^{\vee} : Z \to \Pi_{(A,X)}(Y)$ in $\mathcal{K}_T(A)$. As noticed above, $X \Rightarrow Y = \Pi_{(A,X)}(Y)$ and thus $(f \circ \pi)^{\vee} : A \times Z \to X \Rightarrow Y$ in $\mathbf{B}$. Hence we take

$$\Lambda(f) \quad = \quad (f \circ \pi)^{\vee} \quad \circ \quad \langle id,\ z_0 \circ ! \rangle \quad : \quad A \ \to \ A \times Z \ \to \ X \Rightarrow Y.$$

(The auxiliary type Z is first used to introduce a dummy variable which is later removed by substituting z_0.)
The validity of the categorical (β)- and (η)-equations follows from computations in the fibres. We shall do (β).

$$\begin{aligned}
& ev \circ \Lambda(f) \times id \\
& \quad = \quad \varepsilon^{(t,X)}_Y \quad \circ \quad \langle\langle !,\ \pi' \rangle,\ \pi \rangle \quad \circ \quad \Lambda(f) \times id \\
& \quad = \quad \varepsilon^{(t,X)}_Y \quad \circ \quad (!_A \times id) \times id \quad \circ \quad \langle id,\ \Lambda(f) \circ \pi \rangle \\
& \quad = \quad (!_A \times id)^*(\varepsilon^{(t,X)}_Y) \quad \circ \quad \langle id,\ \Lambda(f) \circ \pi \rangle \\
& \quad = \quad \varepsilon^{(A,X)}_Y \quad \circ \quad \langle id,\ \Lambda(f) \circ \pi \rangle \\
& \quad = \quad \varepsilon^{(A,X)}_Y \quad \circ \quad \langle id,\ \Pi_{(A,X)}(f \circ \pi) \ \circ \ \langle \pi, \eta^{(A,X)}_Z \rangle \ \circ \ \langle id,\ z_0 \circ ! \rangle \ \circ \ \pi \rangle \\
& \quad = \quad \varepsilon^{(A,X)}_Y \ \circ \ \langle \pi,\ \Pi_{(A,X)}(f \circ \pi) \circ \pi \times id \rangle \ \circ \ \langle \pi,\ \eta^{(A,X)}_Z \circ \pi \times id \rangle \ \circ \ \langle id,\ z_0 \circ ! \rangle \\
& \quad = \quad \varepsilon^{(A,X)}_Y \quad \bullet \quad \pi^*_{A,X}\Pi_{(A,X)}(f \circ \pi) \quad \bullet \quad \pi^*_{A,X}(\eta^{(A,X)}_Z) \quad \circ \quad \langle id,\ z_0 \circ ! \rangle
\end{aligned}$$

$$
\begin{aligned}
&= (f \circ \pi) \bullet \varepsilon^{(A,X)}_{\pi^*_{A,X}(Z)} \bullet \pi^*_{A,X}(\eta^{(A,X)}_Z) \circ \langle id,\, z_0 \circ !\rangle \\
&= (f \circ \pi) \bullet id \circ \langle id,\, z_0 \circ !\rangle \\
&= f \circ \pi \circ \langle id,\, z_0 \circ !\rangle \\
&= f. \qquad \square
\end{aligned}
$$

Remark 3.4 The previous lemma gives a reasonably clear picture of what a $\lambda 1$-category is about. It gives rise to some further points.

(i) One obtains an alternative characterization of CCC's: a category $\mathbf{B}$ with finite products is a CCC if and only if $\mathcal{K}_{\mathbf{B}}$ is a $\lambda 1$-category.

(ii) If $(\mathbf{B}, T)$ is a $\lambda 1$-category, then T is enriched over $\mathbf{B}$.

(iii) We describe the often used set theoretic modelling (see [4]) of a simply typed λ-calculus $\lambda 1(C)$ in terms of $\lambda 1$-categories. Suppose that for each type constant $c \in C$ one has a set D_c. Put $D_{\sigma \to \tau} = D_\tau^{D_\sigma}$, the set of functions from D_σ to D_τ. One forms a base category $\mathbf{B}$ with sequences of types as objects. A morphism $\langle \sigma_1, \ldots, \sigma_n \rangle \to \langle \tau_1, \ldots, \tau_m \rangle$ in $\mathbf{B}$ is a sequence $\langle f_1, \ldots, f_m \rangle$ of functions $f_i : D_{\sigma_1} \times \cdots \times D_{\sigma_n} \to D_{\tau_i}$. Composition is done in the obvious way. The empty sequence is terminal and concatenation yields cartesian products. As types take $\mathit{Type}_C \subseteq \mathit{Obj}\,\mathbf{B}$. It obviously yields a $\lambda 1$-category with evaluation $\langle \sigma \to \tau, \sigma \rangle \to \tau$ defined by $(\varphi, a) \mapsto \varphi(a)$ and for $f : \langle \vec{\mu}, \sigma \rangle \to \tau$ the abstraction map $\Lambda(f) : \vec{\mu} \to (\sigma \to \tau)$ described by $\Lambda(f)(\vec{x}) = \boldsymbol{\lambda} y \in D_\sigma.\, f(\vec{x}, y)$.

The above elementary description of $\lambda 1$-categories will be extended to morphisms of these.

Lemma 3.5 *Let $(\mathbf{A}, S)$ and $(\mathbf{B}, T)$ be $\lambda 1$-categories and $H : (\mathbf{A}, S) \to (\mathbf{B}, T)$ be a morphism of CT-structures. One has that H is a morphism of $\lambda 1$-categories iff H preserves the exponents in S (i.e. for $X, Y \in S$ one has $H(X \Rightarrow Y) = HX \Rightarrow HY$ and $H(ev_{X,Y}) = ev_{HX,HY}$).*

Proof : Straightforward. $\square$

Corollary 3.6 *The assignment $\mathbf{B} \mapsto \mathcal{K}_{\mathbf{B}}$ extends to a functor* **CCC** $\to$ **$\boldsymbol{\lambda}$1-cat**. $\square$

The next result is similar to lemma 3.3. It shows that — quite independently from exponent types — cartesian product types are described by left adjoints to weakening functors.

Lemma 3.7 *Let $(\mathbf{B}, T)$ be a CT-structure. Then $\mathcal{K}_T$ has sums if and only if T is closed under cartesian products.*

Proof : The if-part is easy: for $A \in \mathbf{B}$ and $X \in T$ define $\Sigma_{(A,X)} : \mathcal{K}_T(A \times X) \to \mathcal{K}_T(A)$ by $Y \mapsto X \times Y$. Then

$$
\begin{aligned}
\mathcal{K}_T(A)\big(\Sigma_{(A,X)}(Y),\ Z\big) &= \mathbf{B}\big(A \times (X \times Y),\ Z\big) \\
&\cong \mathbf{B}\big((A \times X) \times Y,\ Z\big) \\
&= \mathcal{K}_T(A \times X)\big(Y,\ \pi^*_{A,X}(Z)\big)
\end{aligned}
$$

The only-if-part requires some computations. For $X, Y \in T$, put $X \& Y = \Sigma_{(t,X)}(Y)$. It suffices to show that the unit $\eta = \eta^{(t,X)}_Y : (t \times X) \times Y \to X \& Y$ in $\mathbf{B}$ is an isomorphism. Therefore we define two new projections $p : X \& Y \to X$ and $p' : X \& Y \to Y$. The counit $\varepsilon = \varepsilon^{(t,X)}_Y : t \times (X \& Y) \to Y$ yields the second projection. The first one is obtained by transposing the obvious arrow $(t \times X) \times Y \to X$. Then

$$
\begin{aligned}
p' \circ \eta &= \varepsilon \circ \langle !, id \rangle \circ \eta \\
&= \varepsilon \circ \langle !, \eta \rangle \\
&= \pi' \qquad \text{by a triangular identity;} \\
p \circ \eta &= (\pi' \circ \pi)^{\wedge} \circ \langle !, id \rangle \circ \eta \\
&= (\pi' \circ \pi)^{\wedge} \circ \pi \times id \circ \langle \pi, \eta \rangle \\
&= \pi^*\big((\pi' \circ \pi)^{\wedge}\big) \bullet \eta \\
&= (\pi' \circ \pi)^{\wedge\vee} \\
&= \pi' \circ \pi.
\end{aligned}
$$

Let's put $\alpha = \langle\langle \pi,\, p \circ \pi' \rangle,\, p' \circ \pi' \rangle : t \times (X \& Y) \to (t \times X) \times Y$. In order to show that $\eta \circ \langle\langle !, p \rangle, p' \rangle = id$, it suffices to establish that $\eta \circ \alpha = \pi' : t \times (X \& Y) \to X \& Y$, *i.e.* that $\eta \circ \alpha$ is the identity in the fibre. The latter follows from $(\eta \circ \alpha)^{\vee} = \eta$. Well,

$$
\begin{aligned}
(\eta \circ \alpha)^{\vee} &= \pi^*(\eta \circ \alpha) \bullet \eta \\
&= \eta \circ \alpha \circ \pi \times id \circ \langle \pi, \eta \rangle \\
&= \eta \circ \langle\langle \pi \circ \pi,\, p \circ \eta \rangle,\, p' \circ \eta \rangle \\
&= \eta \circ \langle\langle \pi \circ \pi,\, \pi' \circ \pi \rangle,\, \pi' \rangle \qquad \text{as proved above} \\
&= \eta. \qquad \square
\end{aligned}
$$

Remark 3.8 Given the functor **CCC** $\to$ **$\boldsymbol{\lambda}$1-cat** in 3.6, one naturally asks for an operation working in the reverse direction. Below, two constructions are described; the second one is based on a remark of Peter Johnstone.

(i) Let $(\mathbf{B}, T)$ be a $\lambda 1$-category. By a "brute force" method, one takes $\widehat{\mathbf{B}}$ to be the free CCC generated by $\mathbf{B}$, see [10, I.4]; it comes equipped with an inclusion functor $\mathbf{B} \to \widehat{\mathbf{B}}$ which preserves the exponents which exist in T up-to-isomorphism (by a uniqueness argument). Since morphisms of $\lambda 1$-categories are required to preserve such exponents on-the-nose (see lemma 3.5), one does not have an adjoint situation. Thus, one might want to adapt the definition of morphisms of $\lambda 1$-categories in such a way that one does obtain an adjointness here.

(ii) Assume $(\mathbf{B}, T)$ is a $\lambda 1$-category again. In the preliminary section we described the category $\overline{T}$ obtained by closing T under finite products. It is a CCC since one can put for the newly added objects

$$\begin{array}{rcl} A \Rightarrow t & = & t \\ t \Rightarrow A & = & A \\ C \Rightarrow (A \times B) & = & (C \Rightarrow A) \times (C \Rightarrow B) \\ (A \times B) \Rightarrow C & = & A \Rightarrow (B \Rightarrow C). \end{array}$$

Like before one does not have an adjoint situation, this time due to the fact that the inclusion $\overline{T} \to \mathbf{B}$ needs not map $\overline{T}$ into T.
After the next example one finds a further discussion of this construction.

Example 3.9 Let D be a cpo. Non-empty closed subsets $I \subseteq D$ (*wrt.* the Scott topology) can be described by the following three requirements. (i) $\perp \in I$; (ii) $x \leq y \in I \Rightarrow x \in I$; (iii) directed $X \subseteq I \Rightarrow \bigsqcup X \in I$. We form a base category $\mathbf{B}$ with finite sequences $\langle I_1, \ldots, I_n \rangle$ (for some $n \in \mathbb{N}$) of such non-empty closed subsets as objects. A morphism $\langle I_1, \ldots, I_n \rangle \to \langle J_1, \ldots, J_m \rangle$ in $\mathbf{B}$ is given by a sequence $\langle f_1, \ldots, f_m \rangle$ of continuous functions $f_i : D^n \to D$ satisfying $f_i[\vec{I}] \subseteq J_i$. The empty sequence in $\mathbf{B}$ is then terminal and concatenation yields cartesian products.
Now let's assume that D is isomorphic to its own space of continuous functions $[D \to D]$, via maps $F : D \to [D \to D]$ and $G : [D \to D] \to D$ satisfying $F \circ G = id$ and $G \circ F = id$. As usal, we write $x \cdot y$ for $F(x)(y)$ and $\lambda x.-$ for $G(\boldsymbol{\lambda} x.-)$. An example of such a cpo is Dana Scott's D_∞. In a standard way one forms an exponent of $I, J \subseteq D$ by $I \Rightarrow J = \{x \in D \mid \forall y \in I.\ x \cdot y \in J\}$. One easily verifies that $I \Rightarrow J$ is non-empty and closed in case I, J are.
Let $T \subseteq Obj\,\mathbf{B}$ be the collection of non-empty closed subsets (*i.e.* the sequences consisting of one set). One obtains a full CT-structure $(\mathbf{B}, T)$. We claim that it is a $\lambda 1$-category; using lemma 3.3 this is readily established: one has $ev : \langle I \Rightarrow J, I \rangle \to J$ by $ev = \boldsymbol{\lambda} xy.\ x \cdot y$. For $f : \langle \vec{K}, I \rangle \to J$ one takes $\Lambda(f)(\vec{x}) = \lambda y.\ f(\vec{x}, y)$.

Having seen the the notion of a $\lambda 1$-category, Peter Johnstone suggested the construction in 3.8 (ii) above and argued that one is thus still working with a

CCC but one considers only some of its exponents. Our position is that the $\lambda 1$-category structure is the more fundamental one: for example, in the above example 3.9 the construction from 3.8 (ii) yields the following exponent in $\overline{T} = \mathbf{B}$.

$$\langle I_1, I_2 \rangle \Rightarrow \langle J_1, J_2, J_3 \rangle \quad = \quad \langle I_1 \Rightarrow (I_2 \Rightarrow J_1), I_1 \Rightarrow (I_2 \Rightarrow J_2), I_1 \Rightarrow (I_2 \Rightarrow J_3) \rangle.$$

We think that it is not obvious that this category $\mathbf{B}$ is a CCC (and that exponents are given in such a way) unless one knows the underlying $\lambda 1$-category structure. In a similar way one has that the category of contexts in the term model example 3.2 is a CCC; there is an exponent of contexts which is obtained from the exponent on types.

The interpretation to be described next proceeds in a completely standard way. It forms a preparation for the subsequent theorem — which is as one would expect.

Suppose $(\mathbf{B}, T)$ is a $\lambda 1$-algebra. We use $\Rightarrow$, ev and $\Lambda(-)$ as described in lemma 3.3. For a non-empty collection of constants C together with a function $C \to T$ one can write out an interpretation of the calculus $\lambda 1(C)$ in $(\mathbf{B}, T)$. This is done as follows. Let's denote the function $C \to T$ by $[\![-]\!]$. It can be extended to all types by putting

$$[\![\sigma \to \tau]\!] \quad = \quad [\![\sigma]\!] \Rightarrow [\![\tau]\!].$$

Hence types are interpreted as elements of T and thus as objects of the fibres of $\mathcal{K}_T$. For a context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ we put

$$[\![\Gamma]\!] \quad = \quad [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$$

A certain order is fixed implicitly. The empty context will be interpreted as the terminal object. A term $\Gamma \vdash M : \sigma$ will be interpreted as an arrow $[\![M]\!]_\Gamma : [\![\Gamma]\!] \to [\![\sigma]\!]$. This is done in three steps.

- Variables. For $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ one uses appropriate projections

$$[\![x_i]\!]_\Gamma \quad = \quad \pi_i \quad : \quad [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \quad \to \quad [\![\sigma_i]\!].$$

- Application. For terms $\Gamma \vdash M : \sigma \to \tau$ and $\Gamma \vdash N : \sigma$ one has $[\![M]\!]_\Gamma : [\![\Gamma]\!] \to [\![\sigma]\!] \Rightarrow [\![\tau]\!]$ and $[\![N]\!]_\Gamma : [\![\Gamma]\!] \to [\![\sigma]\!]$. Hence we put

$$[\![MN]\!]_\Gamma \quad = \quad ev \circ \langle [\![M]\!]_\Gamma, [\![N]\!]_\Gamma \rangle \quad : \quad [\![\Gamma]\!] \quad \to \quad [\![\tau]\!].$$

- Abstraction. For $\Gamma, x : \sigma \vdash M : \tau$ one has $[\![M]\!]_{\Gamma, x:\sigma} : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\tau]\!]$. Thus one takes

$$[\![\lambda x : \sigma.M]\!]_\Gamma \quad = \quad \Lambda([\![M]\!]_{\Gamma, x:\sigma}) \quad : \quad [\![\Gamma]\!] \quad \to \quad [\![\sigma]\!] \Rightarrow [\![\tau]\!].$$

Notice that a closed term $M{:}\sigma$ becomes an arrow $[\![M]\!] : t \to [\![\sigma]\!]$.
In order to show that everything works well, one needs the following weakening and substitution lemmas.

(a) For $\Gamma \vdash M{:}\sigma$ with $y{:}\tau \notin \Gamma$ one has

$$[\![M]\!]_{\Gamma, y:\tau} \;=\; [\![M]\!]_\Gamma \circ \pi.$$

(b) For $\Gamma, y{:}\tau \vdash M{:}\sigma$ and $\Gamma \vdash N{:}\tau$ one has

$$[\![M[x := N]]\!]_\Gamma \;=\; [\![M]\!]_{\Gamma, y:\tau} \circ \langle id, [\![N]\!]_\Gamma \rangle.$$

Proofs are by induction on the structure of M. Finally we come to the validity of the (β)- and (η)-conversions.

$$\begin{array}{rcll}
[\![(\lambda x{:}\sigma.M)N]\!]_\Gamma & = & ev \circ \langle [\![\lambda x{:}\sigma.M]\!]_\Gamma,\ [\![N]\!]_\Gamma \rangle & \\
& = & ev \circ \langle \Lambda([\![M]\!]_{\Gamma, x:\sigma}),\ [\![N]\!]_\Gamma \rangle & \\
& = & ev \circ \Lambda([\![M]\!]_{\Gamma, x:\sigma}) \times id \circ \langle id,\ [\![N]\!]_\Gamma \rangle & \\
& = & [\![M]\!]_{\Gamma, x:\sigma} \circ \langle id,\ [\![N]\!]_\Gamma \rangle & \\
& = & [\![M[x := N]]\!]_\Gamma & \text{by the above substitution result.} \\
[\![\lambda x{:}\sigma.Mx]\!] & = & \Lambda([\![Mx]\!]_{\Gamma, x:\sigma}) & \\
& = & \Lambda(ev \circ \langle [\![M]\!]_{\Gamma, x:\sigma},\ [\![x]\!]_{\Gamma, x:\sigma} \rangle) & \\
& = & \Lambda(ev \circ \langle [\![M]\!]_\Gamma \circ \pi,\ \pi' \rangle) & \text{by the above weakening result} \\
& = & \Lambda(ev \circ [\![M]\!]_\Gamma \times id) & \\
& = & [\![M]\!]_\Gamma. &
\end{array}$$

Theorem 3.10 *Let $\mathbf{Sets}_{\neq\emptyset}$ be the category of non-empty sets. There is a forgetful functor $\boldsymbol{\lambda}\mathbf{1}\text{-}\mathbf{cat} \to \mathbf{Sets}_{\neq\emptyset}$ given by $(\mathbf{B}, T) \mapsto T$; it maps a $\lambda 1$-category to its collection of types.*
The assignment $C \mapsto$ [the term model of $\lambda 1(C)$] from example 3.2 yields a left adjoint. Hence the term model of $\lambda 1(C)$ is the free *$\lambda 1$-category generated by C.*

Proof : We have to establish a bijective correpondence

$$\frac{C \longrightarrow T \quad \text{in } \mathbf{Sets}_{\neq\emptyset}}{[\text{term model of } \lambda 1(C)] \longrightarrow (\mathbf{B}, T) \quad \text{in } \boldsymbol{\lambda}\mathbf{1}\text{-}\mathbf{cat}}$$

The passage downwards is essentially given by the interpretation described above. The substitution lemma mentioned there yields the required functoriality. Upwards, one obtains a function $C \to T$ by restricting the interpretation of types to constants.
Notice that the unit of the adjunction inserts a constant $c \in C$ in the set of types of the calculus $\lambda 1(C)$. □

Remark 3.11 In the proof above we cheated a bit. The functor from the term model of $\lambda 1(C)$ to $(\mathbf{B}, T)$ does not preserve cartesian products in the bases "on-the-nose": there is only an isomorphism $[\![\Gamma \times \Delta]\!] \cong [\![\Gamma]\!] \times [\![\Delta]\!]$ instead of an identity. By viewing contexts as appropriate trees, this problem can be avoided. We chose not to do so because it would make the notation rather heavy and distract from the essentials.

In the rest of this section we consider an alternative notion of model without finite product types.

A *multicategory* is given by a set of objects and a set of morphisms. Unlike in ordinary category theory, one allows domains of morphisms to be (possibly empty) sequences of objects. One writes

$$A_1, \ldots, A_n \xrightarrow{f} B.$$

Special morphisms of the following form are required.

$$\begin{array}{rcll}
A & \xrightarrow{id} & A & \quad\text{identity} \\
A_1, \ldots, A_n & \xrightarrow{\pi_i} & A_i & \quad\text{projection}
\end{array}$$

Composition is done in the obvious way: for $f : \langle A_1, \ldots, A_n \rangle \to B$ and $g_i : \vec{C} \to A_i$ for $1 \leq i \leq n$ one has $f \circ (g_1, \ldots, g_n) : \vec{C} \to B$. Straightforward identity and associativity laws should be satisfied. Further one has

$$\begin{array}{rcl}
\pi_i \circ (g_1, \ldots, g_n) & = & g_i \\
f \circ (\pi_1, \ldots, \pi_n) & = & f.
\end{array}$$

A functor $H : \mathbf{M}_1 \to \mathbf{M}_2$ between multicategories is given by two maps $Obj\,\mathbf{M}_1 \to Obj\,\mathbf{M}_2$ and $Mor\,\mathbf{M}_1 \to Mor\,\mathbf{M}_2$, both denoted by H, such that domain, codomain, identities, composition and projections are preserved. For example, one requires

$$A_1, \ldots, A_n \xrightarrow{f} B \text{ in } \mathbf{M}_1 \quad\Rightarrow\quad HA_1, \ldots, HA_n \xrightarrow{Hf} HB \text{ in } \mathbf{M}_2.$$

In [3], a multicategory is called *closed* if for each pair of objects A, B an exponent object $A \Rightarrow B$ exists together with an evaluation map $ev : A \Rightarrow B, A \to B$ such that for each $f : A_1, \ldots, A_n, A \to B$ there is a unique $\Lambda(f) : A_1, \ldots, A_n \to A \Rightarrow B$ such that

$$ev \circ (\Lambda(f) \circ (\pi_1, \ldots \pi_n),\ \pi_{n+1}) \;=\; f.$$

Let **CMC** denote the category of closed multicategories; a morphism $H : \mathbf{M}_1 \to \mathbf{M}_2$ in **CMC** is a functor of multicategories which preserves exponents, *i.e.* satisfies $H(A \Rightarrow B) = HA \Rightarrow HB$ and $H(ev_{A,B}) = ev_{HA,HB}$.

Theorem 3.12 *There is a "forgetful" functor* $\boldsymbol{\lambda}$**1-cat** $\to$ **CMC**; *it has a left adjoint.*

Proof : The functor $U : \boldsymbol{\lambda}\mathbf{1\text{-}cat} \to \mathbf{CMC}$ will be described first. A λ1-category $(\mathbf{B}, T)$ determines a multicategory with $X \in T$ as objects; a morphism f from $X_1, \ldots, X_n$ to Y is simply a map $f : X_1 \times \cdots \times X_n \to Y$ in $\mathbf{B}$. The resulting multicategory is obviously closed, see lemma 3.3. A morphism H of λ1-categories yields a morphism UH of closed multicategories since it preserves cartesian products and exponents on-the-nose.
A functor $F : \mathbf{CMC} \to \boldsymbol{\lambda}\mathbf{1\text{-}cat}$ is obtained by assigning to a closed multicategory $\mathbf{M}$ a category having sequences $A_1, \ldots, A_n$ of objects from $\mathbf{M}$. A morphism $A_1, \ldots, A_n \to B_1, \ldots, B_m$ between such sequences is given by an m-tuple $(f_1, \ldots, f_m)$ of maps $f_i : A_1, \ldots, A_n \to B_i$ in $\mathbf{M}$. The resulting category has finite products: the empty sequence is terminal and concatenation yields cartesian products. As collection of types we take $Obj\,\mathbf{M}$. By definition of closedness for multicategories, it is closed under exponents. Hence one obtains a λ1-category.
There is a natural bijective correspondence

$$\frac{F\mathbf{M} \xrightarrow{H} (\mathbf{B}, T) \quad \text{in } \boldsymbol{\lambda}\mathbf{1\text{-}cat}}{\mathbf{M} \xrightarrow{K} U(\mathbf{B}, T) \quad \text{in } \mathbf{CMC}.}$$

One takes $H^{\vee}(A) = HA$ and $H^{\vee}(f : \vec{A} \to B) = Hf : H\vec{A} \to HB$ using that H preserves finite products. Further, $K^{\wedge}(A_1, \ldots, A_n) = KA_1 \times \cdots \times KA_n$ and $K^{\wedge}(f_1, \ldots, f_m) = \langle Kf_1, \ldots, Kf_m \rangle$. □

4. The untyped λ-calculus

As mentioned in the preliminary section we seek to capture non-extensional models of the type-free λ-calculus. This will be done by using "semi-notions". Restriction to the monoid case gives appropriate untyped settings.

Definition 4.1 *A* categorical λ-algebra *is given by a cartesian category* $\mathbf{B}$ *containing a distinguished object* Ω *such that*
(a) Ω *is a non-empty object, i.e.* $\mathbf{B}(t, \Omega) \neq \emptyset$;
(b) $\mathcal{K}_\Omega : \mathbf{B}^{op} \to \mathbf{Cat}$ *has semi-products.*

In point (b) we mean semi-Ω-products, see the convention at the end of the preliminary section. The first point expresses that the monoid CT-structure $(\mathbf{B}, \{\Omega\})$ is non-trivial. A categorical λ-algebra will be called full if this underlying CT-structure is full.

In order to avoid problems like in remark 3.11 we shall require that appropriate morphisms preserve only specific finite products of the form $A \leftarrow A \times \Omega \to \Omega$ and $A \to t$. This matter will be of relevance in the proof of theorem 4.10.

Definition 4.2 *A* morphism of categorical λ-algebras *from* $(\mathbf{B}, \Omega)$ *to* $(\mathbf{B}', \Omega')$ *is a functor* $H : \mathbf{B} \to \mathbf{B}'$ *satisfying*

- $H\Omega = \Omega'$ *and* $Ht = t'$*, the terminal object in* $\mathbf{B}'$; *moreover,* $H(!_A) = !_{HA}$;
- *for every* $A \in \mathbf{B}$ *one has* $H(A \times \Omega) = (HA) \times \Omega'$ *with* $H(\pi_{A,\Omega}) = \pi_{HA,\Omega'}$ *and* $H(\pi'_{A,\Omega}) = \pi'_{HA,\Omega'}$;
- *the pair* $(H, \widetilde{H})$ *is a morphism* $\mathcal{K}_\Omega \to \mathcal{K}_{\Omega'}$ *of semi-products.*

This yields a category **Cat-$\boldsymbol{\lambda}$-Alg**.

Finally we introduce the usual ramifications.

Definition 4.3 *Let* $(\mathbf{B}, \Omega)$ *be a categorical* λ*-algebra. It will be called*
(a) *a* categorical λ-model *if* Ω *has enough points, i.e. if for* $f, g : \Omega \to A$,

$$\forall x : t \to \Omega.\ f \circ x = g \circ x \quad \Rightarrow \quad f = g;$$

(b) *a* categorical $\lambda\eta$-algebra *if* $\mathcal{K}_\Omega$ *has ordinary products;*
(c) *a* categorical $\lambda\eta$-model *if it is both a categorical* λ*-model and a* $\lambda\eta$*-algebra.*

Instead of describing examples in terms of semi-adjunctions, we first introduce the following two lemmas which give a more elementary description. The proofs are as in the previous section; they can also be found in [6].

Lemma 4.4 *Let* $\mathbf{B}$ *be a category with finite products and* $\Omega \in \mathbf{B}$ *be non-empty. Then*
(a) $(\mathbf{B}, \Omega)$ *is a categorical* λ*-algebra if and only if there is a map*

$$app \;:\; \Omega \times \Omega \;\to\; \Omega$$

together with an operation

$$\lambda(-) \;:\; \mathbf{B}(A \times \Omega, \Omega) \;\to\; \mathbf{B}(A, \Omega)$$

such that

$$\begin{aligned} app \circ \lambda(f) \times id &= f \\ \lambda(f \circ g \times id) &= \lambda(f) \circ g. \end{aligned}$$

(b) $(\mathbf{B}, \Omega)$ *is a categorical* $\lambda\eta$*-algebra if and only if there are app and* λ *as in* (a) *which additionally satisfy*

$$\lambda(app) = id. \qquad \square$$

Lemma 4.5 *Let* $(\mathbf{B}, \Omega)$ *and* $(\mathbf{B}', \Omega')$ *be categorical* λ*-algebras. A functor* $H : \mathbf{B} \to \mathbf{B}'$ *is a morphism of categorical* λ*-algebras if and only if*

- $H\Omega = \Omega'$ *and* $H(!_A) = !_{HA}$;
- $H(\pi_{A,\Omega}) = \pi_{HA,\Omega'}$ *and* $H(\pi'_{A,\Omega}) = \pi'_{HA,\Omega'}$;
- $H(app) = app'$ *and* $H(\lambda(f)) = \lambda'(Hf)$. $\square$

Lemma 4.4 shows that the above notion of categorical λ-algebra is slightly more general then the notion used by Obtulowicz. We recall the latter in the present terminology.

Definition 4.6 (see [11, 12]) *A* Church algebraic theory *is a categorical* λ*-algebra* $(\mathbf{B}, \Omega)$ *where* $\mathbf{B}$ *is an algebraic theory.*
In particular, a Church algebraic theory is a full categorical λ*-algebra.*

The next result is essentially theorem 5.7 in [12].

Proposition 4.7 *If* $(\mathbf{B}, \Omega)$ *is a full categorical* $\lambda\eta$*-algebra, then* $\mathbf{B}$ *is a CCC.*

Proof : By the construction in 3.8 (ii), the category $\overline{\{\Omega\}}$ is a CCC. Hence $\mathbf{B} \simeq \overline{\{\Omega\}}$ as well. $\square$

Next we turn to some examples.

Examples 4.8 (i) Let D be a reflexive cpo via maps $F : D \to [D \to D]$ and $G : [D \to D] \to D$ with $F \circ G = id$, see [1], 5.4. As before we write $a \cdot b$ for $F(a)(b)$ and $\lambda x.-$ for $G(\boldsymbol{\lambda} x.-)$.
A base category $\mathbf{D}$ is formed with $n \in \mathbb{N}$ as objects; n can be considered as the context containing the first n variables from an enumeration $\{x_n \mid n \in \mathbb{N}\}$. Morphism $n \to m$ are sequences $(f_1, \ldots, f_m)$ where each f_i is a continuous function $D^n \to D$, *i.e.* $f_i \in [D^n \to D]$. Composition in $\mathbf{D}$ is done in the obvious way and identities are sequences of projections. The object $0 \in \mathbf{D}$ is terminal and $n + m$ is a product. Thus $\mathbf{D}$ is an algebraic theory. As distinguished object ("Ω") we take $1 \in \mathbf{D}$. Notice that 1 is a non-empty object iff the set D is non-empty. We show that $(\mathbf{D}, 1)$ is a Church algebraic theory.
One has $app : 1 + 1 \to 1$ as a continuous function $D \times D \to D$ described by $(x, y) \mapsto x \cdot y$. For $f : n + 1 \to 1$ in $\mathbf{D}$ one has $\lambda(f)(\vec{x}) = \lambda y.\, f(\vec{x}, y)$. Then

$$\begin{aligned} \big(app \circ \lambda(f) \times id\big)(\vec{x}, z) &= \big(\lambda y.\, f(\vec{x}, y)\big) \cdot z \\ &= f(\vec{x}, z). \end{aligned}$$

The other equation from lemma 4.4 is left to the reader.
One easily verifies that $(\mathbf{D}, 1)$ is a categorical λ-model. In case $G \circ F = id$ — *i.e.* $D \cong [D \to D]$ — it becomes a categorical $\lambda\eta$-model.

(ii) Let $\mathsf{M} = \langle D, \cdot, \boldsymbol{K}, \boldsymbol{S} \rangle$ be a (set theoretic) λ-algebra, see [1, 5.2]. One writes $\mathbf{1}_n = \lambda x_0 \ldots x_n.\, x_0 \ldots x_n$; inductively, one can define $\mathbf{1}_0 = \boldsymbol{I} = \boldsymbol{SKK}$ and $\mathbf{1}_{n+1} = \boldsymbol{S}(\boldsymbol{K}\mathbf{1}_n)$, see *loc. cit.* 5.6. Let's put $(D^n \to D) = \{a \in D \mid \mathbf{1}_n \cdot a = a\}$. Then $(D^0 \to D) = D$; we write $\mathbf{1}$ for $\mathbf{1}_1$ and $(D \to D)$ for $(D^1 \to D)$.
Let $\mathbf{D}$ be a base category, once again with $n \in \mathbb{N}$ as objects, but with m-tuples $(a_1, \ldots, a_m)$ with $a_i \in (D^n \to D)$ as morphisms $n \to m$. Then $(b_1, \ldots, b_k) \circ (a_1, \ldots, a_m) = (c_1, \ldots, c_k)$ where c_i is the term $\lambda x_1 \ldots x_n.\, b_i(a_1 x_1 \ldots x_n) \ldots (a_m x_1 \ldots x_n)$. The identity on n is $(\lambda x_1 \ldots x_n.\, x_1, \ldots, \lambda x_1 \ldots x_n.\, x_n)$. The category $\mathbf{D}$ has terminal 0 and products $n + m$ as before. Hence it is an algebraic theory again. We take $1 \in \mathbf{D}$ as distinguished object.
Lemma 4.4 will be used to show that $(\mathbf{D}, 1)$ is a categorical λ-algebra. One has $app = \lambda xy.\, xy \in (D^2 \to D)$. For $a \in (D^{n+1} \to D)$ one takes $\lambda(a) = a$, which is in $(D^n \to D)$. Then

$$\begin{aligned} ev \circ \lambda(a) \times id &= \lambda x_1 \ldots x_{n+1}.\, (a \cdot x_1 \cdot \ldots \cdot x_n) \cdot x_{n+1} \\ &= \lambda x_1 \ldots x_{n+1}.\, a \cdot x_1 \cdot \ldots \cdot x_n \cdot x_{n+1} \\ &= \mathbf{1}_{n+1} \cdot a \\ &= a. \end{aligned}$$

In case M is a λ-model, *i.e.* $\forall x \in D.\ a \cdot x = b \cdot x \ \Rightarrow\ \mathbf{1} \cdot a = \mathbf{1} \cdot b$, one obtains a categorical λ-model: suppose morphisms $(a_1, \ldots, a_m), (b_1, \ldots, b_m) : 1 \to m$ in $\mathbf{D}$ are given with $\forall x : 0 \to 1.\ (a_1, \ldots, a_m) \circ x = (b_1, \ldots, b_m) \circ x$. Then $a_i, b_i \in (D \to D)$ satisfy $\forall x \in D.\ a_i \cdot x = b_i \cdot x$. Hence $a_i = \mathbf{1} \cdot a_i = \mathbf{1} \cdot b_i = b_i$. Thus the object $1 \in \mathbf{D}$ has enough points.

(iii) Suppose $\mathbf{B}$ is a CCC which has a reflexive object Ω. The latter means that there are maps $F : \Omega \to (\Omega \Rightarrow \Omega)$ and $G : (\Omega \Rightarrow \Omega) \to \Omega$ with $F \circ G = id$. Such structures are used by Scott and Koymans for the semantics of the untyped λ-calculus, see [13, 9]. Using lemma 4.4 one easily obtains a categorical λ-algebra $(\mathbf{B}, \Omega)$; one defines

$$\begin{array}{rcl} app & = & ev \circ F \times id \\ \lambda(f) & = & G \circ \Lambda(f). \end{array}$$

This yields the required equations.

$$\begin{array}{rcl} app \ \circ \ \lambda(f) \times id & = & ev \ \circ \ F \times id \ \circ \ (G \circ \Lambda(f)) \times id \\ & = & ev \ \circ \ \Lambda(f) \times id \\ & = & f. \\ \lambda(f \circ g \times id) & = & G \ \circ \ \Lambda(f \circ g \times id) \\ & = & G \ \circ \ \Lambda(f) \ \circ \ g \\ & = & \lambda(f) \ \circ \ g. \end{array}$$

Moreover, in case $(\mathbf{B}, \Omega)$ is extensional in the sense of Scott and Koymans — which means that $G \circ F = id$ and thus $(\Omega \Rightarrow \Omega) \cong \Omega$ — then

$$\begin{array}{rcl} \lambda(app) & = & G \ \circ \ \Lambda(ev \circ F \times id) \\ & = & G \ \circ \ F \\ & = & id. \end{array}$$

Remark 4.9 (i) A categorical λ-algebra as it is used here is "more economical" than the structure used by Scott and Koymans: in our case there is no requirement about exponents in the base category $\mathbf{B}$.

(ii) The reflexive domain D used in 4.8 (i) is a reflexive object in the category of cpo's with continuous functions. It yields a non-full CT-structure and hence one does not have an algebraic theory this time.

(iii) The conditions in lemma 4.4 (i) essentially express that $(\mathbf{B}, \Omega)$ is a categorical λ-algebra if and only if the functor $\mathbf{B}(-, \Omega)$ is a reflexive object in the topos of presheaves $\mathbf{Sets}^{\mathbf{B}^{op}}$. This is because

$$\begin{array}{rcl} \big(\mathbf{B}(-, \Omega) \Rightarrow \mathbf{B}(-, \Omega)\big)(A) & = & Nat\big(\mathbf{B}(-, A) \times \mathbf{B}(-, \Omega),\ \mathbf{B}(-, \Omega)\big) \\ & \cong & Nat\big(\mathbf{B}(-, A \times \Omega),\ \mathbf{B}(-, \Omega)\big) \\ & \cong & \mathbf{B}(A \times \Omega,\ \Omega). \end{array}$$

At the end of this section one finds another way to obtain a CCC with a reflexive object from a categorical λ-algebra.

Let $(\mathbf{B}, \Omega)$ be a categorical λ-algebra. For $a, b \in \mathbf{B}(A, \Omega)$ put $a \cdot b = app \circ \langle a, b \rangle$. We write $\|\Omega\|$ for the (non-empty) collection $\mathbf{B}(t, \Omega)$ and claim that

$\langle \|\Omega\|, \cdot \rangle$ is a (set theoretic) λ-algebra as described in [1]. Abstraction is done as follows. For a term $a(x) : t \times \Omega \to \Omega$ containing a free variable x one takes

$$\lambda x.\, a(x) \ = \ \lambda(a(x)) \ : \ t \ \to \ \Omega.$$

Then

$$\begin{array}{rcl} \big(\lambda x.\, a(x)\big) \cdot b & = & app \ \circ \ \langle \lambda(a(x)),\, b \rangle \\ & = & app \ \circ \ \lambda(a(x)) \times id \ \circ \ \langle id, b \rangle \\ & = & a(x) \ \circ \ \langle id, b \rangle \\ & = & a(b). \end{array}$$

Let's write

$$\pi_i^n \ : \ t \times \underbrace{\Omega \times \cdots \times \Omega}_{n \text{ times}} \ \to \ \Omega$$

for the i-the projection. One has

$$\begin{array}{rcl} \boldsymbol{K} & = & \lambda(\lambda(\pi_1^2)) \\ \boldsymbol{S} & = & \lambda(\lambda(\lambda(\,(\pi_1^3 \cdot \pi_3^3) \ \cdot \ (\pi_2^3 \cdot \pi_3^3)\,))) \\ \boldsymbol{I} & = & \lambda(\pi_1^1) \\ \mathbf{1} & = & \lambda(\lambda(\,\pi_1^2 \cdot \pi_2^2\,)) \end{array}$$

which yields essentially de Bruijn's nameless notation.

Notice that for $a \in \|\Omega\|$ one has $\mathbf{1} \cdot a = \lambda y.\, a \cdot y = \lambda(app \circ a \times id)$. Hence if $(\mathbf{B}, \Omega)$ is a categorical λ-model, one obtains the (ξ)-rule.

$$\begin{array}{ll} \forall x \in \|\Omega\|.\ a \cdot x = b \cdot x & \\ \quad \Rightarrow \quad \forall x : t \to \Omega.\ app \circ a \times id \circ \langle id,\, x \rangle = app \circ b \times id \circ \langle id,\, x \rangle & \\ \quad \Rightarrow \quad app \circ a \times id = app \circ b \times id, & \text{since } t \times \Omega \cong \Omega \text{ has enough points} \\ \quad \Rightarrow \quad \mathbf{1} \cdot a = \mathbf{1} \cdot b. & \end{array}$$

And if $(\mathbf{B}, \Omega)$ is a categorical $\lambda\eta$-algebra, then (η) holds.

$$\begin{array}{rcl} \lambda y.\, a \cdot y & = & \lambda(app \circ a \times id) \\ & = & \lambda(app) \ \circ \ a \\ & = & a. \end{array}$$

We write **Set-$\boldsymbol{\lambda}$-Alg** for the category with (set theoretic) λ-algebras $\langle D, \cdot, \boldsymbol{K}, \boldsymbol{S} \rangle$ as objects; we allow D to be a collection of arbitrary size. Morphisms are maps between the underlying collections preserving application and $\boldsymbol{K}, \boldsymbol{S}$, see [1, 5.2.2 (ii)].

The assignment $(\mathbf{B}, \Omega) \mapsto \langle \|\Omega\|, \cdot \rangle$ forms the object-part of a "forgetful" functor $U : \mathbf{Cat\text{-}\lambda\text{-}Alg} \longrightarrow \mathbf{Set\text{-}\lambda\text{-}Alg}$: for a morphism $K : (\mathbf{B}, \Omega) \to (\mathbf{B}', \Omega')$ of categorical λ-algebras, one has $UK : \|\Omega\| \to \|\Omega'\|$ defined by $a \mapsto Ka$. By lemma 4.5, K preserves *app* and λ on-the-nose; hence UK is a morphism of λ-algebras.

Theorem 4.10 *The forgetful functor* $U : \mathbf{Cat\text{-}\lambda\text{-}Alg} \longrightarrow \mathbf{Set\text{-}\lambda\text{-}Alg}$ *has a left adjoint; the unit of the adjunction is an identity.*
The adjointness gives rise to an equivalence between $\mathbf{Set\text{-}\lambda\text{-}Alg}$ *and the full subcategory of* $\mathbf{Cat\text{-}\lambda\text{-}Alg}$ *of Church algebraic theories.*

Proof : The object-part of a functor $F : \mathbf{Set\text{-}\lambda\text{-}Alg} \longrightarrow \mathbf{Cat\text{-}\lambda\text{-}Alg}$ is described in example 4.8 (ii). For a morphism of λ-algebras $h : \langle D, \cdot \rangle \longrightarrow \langle D', \cdot' \rangle$ one defines $Fh : (\mathbf{D}, 1) \to (\mathbf{D}', 1)$ by $n \mapsto n$ and $(a_1, \ldots, a_m) \mapsto (h(a_1), \ldots, h(a_m))$. By lemma 4.5 and proposition 5.1.14 (i) from [1], h preserves the relevant structure. Notice that the underlying collection of $UF(\langle D, \cdot \rangle)$ is $\|1\| = \mathbf{D}(0, 1) = (D^0 \to D) = D$. One obtains $UF = id$.
A counit $\varepsilon : FU(\mathbf{B}, \Omega) \to (\mathbf{B}, \Omega)$ is defined on objects by $n \mapsto \Omega^n$. To define it on morphisms, we need some notation. For an element $a \in \|\Omega\|$ we define $a^{(n)} : \Omega^n \to \Omega$ by $a^{(n)} = (a \circ !_{\Omega^n}) \cdot \pi_1^n \cdot \ldots \cdot \pi_n^n$ where $\pi_i^n : \Omega^n \to \Omega$ is the i-th projection.
On a morphism $(a_1, \ldots, a_m) : n \to m$ in $FU(\mathbf{B}, \Omega)$ — where $a_i \in (\|\Omega\|^n \to \|\Omega\|)$ — we put $\varepsilon(a_1, \ldots, a_m) = \langle a_1^{(n)}, \ldots, a_m^{(n)} \rangle : \Omega^n \to \Omega^m$. One has

$$\varepsilon(\lambda x_1 \ldots x_n . x_i) \;=\; ((\lambda x_1 \ldots x_n . x_i) \circ !) \cdot \pi_1^n \cdot \ldots \cdot \pi_n^n \;=\; \pi_i^n.$$

Hence ε preserves identities and the projections $n \leftarrow n+1 \to 1$. Composition is preserved since

$$\varepsilon(\lambda x_1 \ldots x_n . b_i(a_1 x_1 \ldots x_n) \ldots (a_m x_1 \ldots x_n)) \;=\; b_i^{(m)} \circ \langle a_1^{(n)}, \ldots, a_m^{(n)} \rangle$$

In order to show that ε is a morphism of categorical λ-algebras it suffices by lemma 4.5 to check

$$\begin{array}{rcll} \varepsilon(app) & = & (\lambda xy. xy)^{(2)} & \text{see 4.8 (ii)} \\ & = & \pi_1^2 \cdot \pi_2^2 & \\ & = & app \circ \langle \pi, \pi' \rangle & \\ & = & app. & \end{array}$$

and for $a \in (\|\Omega\|^{n+1} \to \|\Omega\|)$,

$$\begin{array}{rcll} \varepsilon(\lambda(a)) & = & \varepsilon(a) & \text{see 4.8 (ii)} \\ & = & (a \circ !) \cdot \pi_1^n \cdot \ldots \cdot \pi_n^n & \\ & = & \lambda x. (a \circ !) \cdot \pi_1^n \cdot \ldots \cdot \pi_n^n \cdot x & \text{since } \mathbf{1}_{n+1} \cdot a = a \\ & = & \lambda(app \circ ((a \circ !) \cdot \pi_1^n \cdot \ldots \cdot \pi_n^n) \times id) & \\ & = & \lambda((a \circ !) \cdot \pi_1^{n+1} \cdot \ldots \cdot \pi_{n+1}^{n+1}) & \\ & = & \lambda(\varepsilon(a)). & \end{array}$$

The triangular identities boil down to

$$\varepsilon F \;=\; id \qquad \text{and} \qquad U\varepsilon \;=\; id.$$

These are easily verified. Finally the counit $\varepsilon : FU(\mathbf{B}, \Omega) \to (\mathbf{B}, \Omega)$ is an isomorphism if and only if $\mathbf{B}$ is an algebraic theory. □

The pattern obtained here is the same as established in [7], 7.4.3 for the second order λ-calculus: the functor from categorical to set theoretical models has a left-adjoint-right-inverse.

Next we deal with some categorical properties of categorical λ-algebras.

Theorem 4.11 *Let* $(\mathbf{B}, \Omega)$ *be a categorical* λ*-algebra. Then*
(a) $\mathcal{K}_\Omega$ *is an indexed monoid, i.e. all fibre categories are monoids;*
(b) $\mathcal{K}_\Omega$ *is an indexed semi-CCC, i.e. all fibre categories are semi-CCC's and reindexing preserves this structure.*

Proof : (a) Obvious, since one starts from a single type Ω.
(b) The standard (non-surjective) pairing from λ-calculus yields "combinators" $fst, snd : \Omega \to \Omega$ and $pair : \Omega \times \Omega \to \Omega$ satisfying $fst \circ pair = \pi$ and $snd \circ pair = \pi'$. In λ-calculus notation, $fst(z) = z\boldsymbol{K}$, $snd(z) = z\boldsymbol{K}'$ — where $\boldsymbol{K}' = \lambda xy. y$ — and $pair(x, y) = \lambda z. zxy$. A bit more categorically, $fst = id_\Omega \cdot (\boldsymbol{K} \circ !_\Omega)$, $snd = id_\Omega \cdot (\boldsymbol{K}' \circ !_\Omega)$ and $pair = \lambda(\pi' \cdot (\pi \circ \pi) \cdot (\pi' \circ \pi))$. Remember from the preliminary section that composition in the fibre categories $\mathcal{K}_\Omega(A)$ is described by $g \bullet f = g \circ \langle \pi, f \rangle$. We define the semi-CCC structure in the fibre categories, see [5].
(1) $!' = c_0 \circ !_{A \times \Omega} : A \times \Omega \to \Omega$, where $c_0 : t \to \Omega$ is an arbitrary constant; then $!' \bullet f = c_0 \circ !_{A \times \Omega} \circ \langle \pi, f \rangle = c_0 \circ !_{A \times \Omega} = !'$.
(2) $\pi_0 = fst \circ \pi'$, $\pi_1 = snd \circ \pi' : A \times \Omega \to \Omega$. Further, for $f, g : A \times \Omega \to \Omega$ one takes $\langle\langle f, g \rangle\rangle = pair \circ \langle f, g \rangle$. Then $\pi_0 \bullet \langle\langle f, g \rangle\rangle = f$, $\pi_1 \bullet \langle\langle f, g \rangle\rangle = g$ and $\langle\langle f, g \rangle\rangle \bullet h = \langle\langle f \bullet h, g \bullet h \rangle\rangle$.
(3) $ev = app \circ \langle fst, snd \rangle \circ \pi' : A \times \Omega \to \Omega$. For $f : A \times \Omega \to \Omega$ one takes $\Lambda(f) = \lambda(f \circ \langle \pi \circ \pi, pair \circ \pi \times id \rangle)$. Then $ev \bullet \langle\langle \Lambda(f) \bullet g, h \rangle\rangle = \Lambda(f) \bullet \langle\langle g, h \rangle\rangle$, $\Lambda(f \bullet \langle\langle g \bullet \pi_0, \pi_1 \rangle\rangle) = \Lambda(f) \bullet g$ and $ev \bullet \langle\langle \pi_0, \pi_1 \rangle\rangle = ev$. □

The previous theorem yields a second construction of a "CCC with reflexive object" (see 4.8 (iii)) from a categorical λ-algebra (the first one is in 4.9). The next two facts should be used.

- The Karoubi envelope $K(\mathbf{C})$ of a semi-CCC $\mathbf{C}$ is a CCC, see [5].
- If $X \cong (X \Rightarrow X)$ in a semi-CCC $\mathbf{C}$, then id_X is a reflexive object in $K(\mathbf{C})$. The latter is easily verified.

Obviously, the object, say X, of a monoid semi-CCC satisfies $X \cong (X \Rightarrow X)$. Hence taking the Karoubi envelope of one of the fibre categories of a categorical λ-algebra $\mathcal{K}_\Omega$ yields a CCC with a reflexive object.

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