Coalgebras and Approximation[1]

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Abstract. *Motivated by a new approach in the categorical semantics of linear logic, we investigate some specific categories of coalgebras. They all arise from the canonical comonad that one has on a category of algebras. We obtain a very simple model of linear logic where linear formulas are complete lattices and intuitionistic formulas are just sets. Also, in another, domain theoretic example, we give a new characterization of continuous posets (where every elemtent is join of elements way below) as coalgebras. And finally we describe a related example where categories in which every object is coproduct of indecomposables, are coalgebras. Approximation is the key ingredient of all these coalgebras.*

1 Introduction

The starting point for our investigation is the work of Benton, Bierman, Hyland and de Paiva [5, 4]. It contains a new description of the intuitionistic subsystem of linear logic. The standard semantic picture (see Seely [12]) displays this subsystem as lying in the Kleisli category of the shriek ! comonad of linear logic. In the new approach, an intuitionistic formula is a coalgebra of the ! comonad: it is a map $\varphi\colon A \to\, !A$, where the underlying object A is a linear formula. Logically, φ can be understood as a proof of $A \vdash\, !A$. Computationally, one can see φ as a program $A \to\, !A$ which can turn an element $a \in A$ into as many copies as one likes. It is a supplier of A's.

The coalgebra semantics is used in [5, 4] to derive a new term calculus for linear logic. But no concrete examples of categories of coalgebras are given. This will be done in the present paper. But linear logic is only a starting point for our research, and does not really play a role in the sequel. Only the first example of the powerset comonad on complete lattices below is of interest in the semantics of linear logic. In the second example we investigate the ideal comonad on directed complete partial orders. It leads to a new characterization of continuous posets, namely as coalgebras of this comonad. This example is of interest in domain theory. Our third example has a categorical interest. It involves the family comonad on the category of categories with arbitrary coproducts.

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[1]The research presented here was partly carried out during a visit to Sydney, supported by the Australian and Dutch research councils ARC and NWO. The paper appeared in: A. Nerode, Yu. V. Matiyasevich (eds.) *Logical Foundations of Computer Science* Springer LNCS 813, 1994, p. 173–183.

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The comonads that we study all arise in the same way (as in [7]): namely as coming from the standard adjunction between a category of algebras and the underlying category. From a purely categorical point of view we can give the following alternative motivation for our work. Consider a category $\mathbb{C}$ with a monad T on it. Then we can form the category of algebras $\mathrm{Alg}(T)$, which comes with a standard adjunction $\mathrm{Alg}(T) \rightleftarrows \mathbb{C}$. This adjunction induces a comonad on $\mathrm{Alg}(T)$ and we can look at its category of coalgebras, with the associated adjunction. This in turn induces a monad on the category of coalgebras, etcetera. One gets a sequence, which starts with the monad T on the right:

$$\begin{array}{ccccccccc}
\cdots & \leftrightarrows & \mathrm{Alg}(T_2) & \rightleftarrows & \mathrm{CoAlg}(T_1) & \leftrightarrows & \mathrm{Alg}(T_0) & \rightleftarrows & \mathbb{C} \\
 & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
 & & T_3 & & T_2 & & T_1 & & T_0 = T \\
 & & & & & & & & \\
 & & \text{comonad} & & \text{monad} & & \text{comonad} & & \text{monad}
\end{array}$$

In principle this sequence can be extended *ad infinitum*. One can ask the following questions.

What kind of structure do these algebras and coalgebras correspond to? Below we concentrate in particular on the second step of the coalgebras in $\mathrm{CoAlg}(T_1)$.

Does such a sequence ever reach a "fixed point"?

In the three examples that we investigate we'll see the following. Starting from $T =$ (powerset monad) on $\mathbb{C} = \mathbf{Sets}$ one reaches a fixed point in two steps: one gets that the 2^{nd} category $\mathrm{CoAlg}(T_1)$ is equivalent to the 0^{th} category $\mathbf{Sets}$. For the ideal monad on posets one gets that the categories $\mathrm{Alg}(T_2)$ and $\mathrm{Alg}(T_0)$ are equivalent. Hence here one has a fixed point in three steps. In our final examples of the family monad on categories we don't know what happens beyond step 2.

2 Coalgebras and complete lattices

In this section we start from the powerset monad $\mathcal{P}$ on the category $\mathbf{Sets}$ of sets and functions. Its category of algebras is the category $\mathbf{CL}$ of complete lattices (posets in which every subset has a join) and linear functions (preserving all these joins), see e.g. [10, VI, 2, Exercise 1] or [8, I, Corollary 4.5]. This category is $*$-autonomous (see [3]). The induced comonad—written as $\mathcal{P}_1$ like in the above sequence—on $\mathbf{CL}$ is based on powerset: it assigns to a complete lattice $(A, \leq)$ the free complete lattice $(\mathcal{P}A, \subseteq)$ of subsets of the underlying set A. It forms in this way a model of linear logic with as Kleisli category the category of complete lattices with ordinary functions between the underlying sets (because a linear function $(\mathcal{P}A, \subseteq) \to (B, \leq)$ can be identified with an ordinary function $A \to B$).

Theorem 2.3 below tells that the category $\mathrm{CoAlg}(\mathcal{P}_1)$ of coalgebras of the powerset comonad $\mathcal{P}_1$ on complete lattices is (equivalent to) $\mathbf{Sets}$. This shows that the above sequences reaches for this example a fixed point in two steps. It also gives a simpler model of linear logic with linear formulas as complete lattices and intuitionistic formulas as sets. The result is obtained by unravelling the structure given by a coalgebra for this comonad. It turns out that such a coalgebra gives an approximation of each element by the atoms below it. And complete lattices with this property are necessarily free (i.e. of the form $\mathcal{P}X$ for a set X). This aspect of approximation can also be found in the other two examples: continuous posets (in which each element is the supremum of the elements way below it) are coalgebras and categories with coproducts in which each object is coproduct of indecomposables are also coalgebras.

This first result (Theorem 2.3) below which characterizes coalgebras of the powerset comonad on complete lattices is a special case of a more general result of Barr [2][3], which is established using a monadicity theorem of Beck. We choose to spell out an elementary proof because it brings the abovementioned aspect of approximation clearly to the surface and thus gives a link with (and preparation for) the other two examples.

For the powerset comonad $\mathcal{P}_1$ on **CL** we have that counits $\varepsilon_A: \mathcal{P}A \to A$ are joins $\bigvee$ and diagonals $\delta_A: \mathcal{P}A \to \mathcal{P}^2 A$ send a subset $a \subseteq A$ to the set of sets $\{\{x\} \mid x \in a\}$.

An element x in a complete lattice B will be called an **atom** if it satisfies

(i) $x \neq \bot$

(ii) $y \leq x \;\Rightarrow\; y = \bot$ or $y = x$

(iii) $x \leq \bigvee_i z_i \;\Rightarrow\; \exists i.\, x \leq z_i$.

We write $\mathcal{A}(B)$ for the set of atoms in B. A complete lattice B will be called **atomic** if for each element $x \in B$ one has

$$x = \bigvee \{y \in \mathcal{A}(B) \mid y \leq x\}.$$

For example, each free complete lattice $\mathcal{P}X$ is atomic with singletons $\{x\} \in \mathcal{P}X$ as atoms.

2.1. Lemma. *Let $\varphi: B \to \mathcal{P}B$ be a coalgebra of the powerset comonad $\mathcal{P}_1$ on* **CL**. *The coalgebra equation $\varepsilon_B \circ \varphi = id$ and $\mathcal{P}(\varphi) \circ \varphi = \delta_B \circ \varphi$ translate into,*

$$\bigvee \varphi(x) = x \qquad \textit{and} \qquad \{\varphi(y) \mid y \in \varphi(x)\} = \{\{y\} \mid y \in \varphi(x)\}.$$

Then,

(i) $\varphi(x) = \emptyset \;\Leftrightarrow\; x = \bot$.

(ii) $y \in \varphi(x) \;\Rightarrow\; y \leq x$ *and* $\varphi(y) = \{y\}$.

(iii) *y is an atom (i.e. $y \in \mathcal{A}(B)$)* $\;\Leftrightarrow\; \varphi(y) = \{y\}$.

(iv) $\varphi(x) = \{y \in \mathcal{A}(B) \mid y \leq x\}$.

Proof. The results are obtained by unravelling the data.

(i) Since φ is a linear function, one has $\varphi(\bot) = \emptyset$. Conversely, if $\varphi(x) = \emptyset$, then $x = \bigvee \varphi(x) = \bigvee \emptyset = \bot$.

[3] With thanks to Peter Johnstone for pointing this out.

(ii) Suppose $y \in \varphi(x)$, then $y \leq \bigvee \varphi(x) = x$. Also, $\varphi(y) = \{y'\}$, for some $y' \in \varphi(x)$, because φ is a coalgebra. But then $y = \bigvee \varphi(y) = \bigvee \{y'\} = y'$ and thus $\varphi(y) = \{y\}$.

(iii) Suppose y is an atom; then $y \neq \perp$ and so $\varphi(y) \neq \emptyset$, by (i). If $z \in \varphi(y)$, then $z \leq y$ and $\varphi(z) = \{z\}$. Hence $z \neq \perp$, and thus $z = y$ (since y is an atom). But then $\varphi(y) = \{y\}$.

Conversely, suppose $\varphi(y) = \{y\}$. By (i) $y \neq \perp$. Further, if $z \leq y$, then $\varphi(z) \subseteq \varphi(y) = \{y\}$ by (ii). Hence $\varphi(z) = \emptyset$ or $\varphi(z) = \{y\}$. In the first case one has $z = \bigvee \varphi(z) = \bigvee \emptyset = \perp$ and in the second case $z = \bigvee \varphi(z) = \bigvee \{y\} = y$.

Finally, assume $y \leq \bigvee_i z_i$. Then $y \in \{y\} = \varphi(y) \subseteq \varphi(\bigvee_i z_i) = \bigcup_i \varphi(z_i)$. Hence for some i, $y \in \varphi(z_i)$, and thus $y \leq z_i$, by (ii).

(iv) The inclusion ($\subseteq$) follows from (ii) and (iii). Conversely, assume y is an atom with $y \leq x$. Then by (iii), $y \in \varphi(y) \subseteq \varphi(x)$. $\square$

2.2. Proposition. *A complete lattice is atomic if and only if it carries a (necessarily unique) coalgebra structure. It is given by $x \mapsto \{y$ atom and $y \leq x\}$.*

Proof. Suppose $\varphi: B \to \mathcal{P}B$ is a coalgebra. Then by (iv) in the previous lemma, $x \in B$ satisfies $x = \bigvee \varphi(x) = \bigvee \{y \in \mathcal{A}(B) \mid y \leq x\}$.

Conversely, if B is an atomic complete lattice, then $\varphi: B \to \mathcal{P}B$ defined by $x \mapsto \{y \in \mathcal{A}(B) \mid y \leq x\}$ is a coalgebra indeed. This is checked easily. $\square$

2.3. Theorem. *The category* $\mathrm{CoAlg}(\mathcal{P}_1)$ *of coalgebras of the powerset comonad* $\mathcal{P}_1$ *on complete lattices is (equivalent to) the category of sets.*

Proof. A morphism of coalgebras $(B \xrightarrow{\varphi} \mathcal{P}B) \xrightarrow{f} (C \xrightarrow{\psi} \mathcal{P}C)$ is a linear function $f: B \to C$ which makes the following diagram commute,

$$\begin{tikzcd} \mathcal{P}B \arrow[r, "\mathcal{P}f"] & \mathcal{P}C \\ B \arrow[u, "\varphi"] \arrow[r, "f"'] & C \arrow[u, "\psi"'] \end{tikzcd}$$

where both φ and ψ take an element to the set of atoms below it. Then, if $y \in B$ is an atom, one has

$$\psi(f(y)) = \mathcal{P}f(\varphi(y)) = \mathcal{P}f(\{y\}) = \{f(y)\}$$

which makes $f(y)$ an atom as well. Hence a coalgebra map $f: B \to C$ is completely determined by its restriction $f: \mathcal{A}(B) \to \mathcal{A}(C)$ to atoms. Put a bit differently, one has a full and faithful functor

$$\mathcal{A}(\_): \mathrm{CoAlg}(\mathcal{P}_1) \longrightarrow \mathbf{Sets}$$

It is an equivalence, since for every set X one has that $\mathcal{A}(\mathcal{P}X) \cong X$. $\square$

3 Coalgebras and directed complete partial orders

Let $X = \langle X, \leq \rangle$ be a poset. An **ideal** in X is a subset $a \subseteq X$ which is a directed downset. We write $\mathrm{Idl}(X)$ for the poset of ideals in X, ordered by inclusion. The assignment $X \mapsto \mathrm{Idl}(X)$ extends to a functor on the category **PoSets** of posets and monotone functions. One maps $f: X \to Y$ to $\mathrm{Idl}(f): \mathrm{Idl}(X) \to \mathrm{Idl}(Y)$ by $a \mapsto \downarrow\{f(x) \mid x \in a\}$. This functor carries the structure of a monad, with unit $\eta: X \to \mathrm{Idl}(X)$ given by $x \mapsto \downarrow x = \{y \in X \mid y \leq x\}$ and multiplication $\mu: \mathrm{Idl}(\mathrm{Idl}(X)) \to \mathrm{Idl}(X)$ by $A \mapsto \bigcup A$. The following is a familiar basic result about this monad. Our main theorem in this section gives in some sense a dual version.

3.1. Proposition. *For a poset X, the following are equivalent.*

(i) *X is a directed complete partial order (dcpo): each directed subset $a \subseteq X$ has a join $\bigvee a$ in X.*

(ii) *The unit $\downarrow(-): X \to \mathrm{Idl}(X)$ has a left adjoint; it is then the join $\bigvee$.*

(iii) *X carries a (necessarily unique) algebra structure $\mathrm{Idl}(X) \to X$, which is also the join.* □

Additionally, one can show that the algebra maps are precisely the Scott-continuous functions (which preserve directed joins). The resulting category will be written as **Dcpo**. Because the join $\bigvee: \mathrm{Idl}(X) \to X$ is an algebra, it is by its nature an algebra map, and thus a continuous function.

The monad $\mathrm{Idl} = \mathrm{Idl}_0$ on **PoSets** induces a comonad, written Idl_1, on **Dcpo**. In order to study its category of coalgebras we need the following. In a dcpo X, the **way below** relation $\ll$ is defined as: for $x, y \in X$,

$$x \ll y a \subseteq X, \text{ if } y \leq \bigvee a \text{ then } \exists z \in a.\, x \leq z$$

A **continuous poset** is then a dcpo in which for each element $x \in X$ the set $\Downarrow x = \{y \in X \mid y \ll x\}$ is directed and has x as join. Continuous posets are given a central position in the survey article [1] on domain theory.

It is easy to see that $x \ll y \Rightarrow x \leq y$. Thus $\ll$ is anti-symmetric. It is obviously transitive. Further, if $x' \leq x \ll y \leq y'$ then $x' \ll y'$. We shall make use below of the following interpolation result, which is a mild generalization of [8, VII, Lemma 2.4]. For each continuous function $f: X \to Y$ between continuous posets X, Y one has

$$y \ll f(x) \Rightarrow \exists z \in X.\, y \ll f(z) \;\&\; z \ll x.$$

The result is easy to prove. Given $x \in X$, consider the subset $b = \bigcup\{\Downarrow f(z) \mid z \in \Downarrow x\}$ which is directed and has $f(x)$ as supremum. Thus, if $y \ll f(x) = \bigvee b$, then $y \leq w$, for some $w \in b$. But then $w \in \Downarrow f(z)$ for some $z \in \Downarrow x$. Hence $y \ll f(z)$ and $z \ll x$.

Our main result is then the following.

3.2. Theorem. *For a dcpo X, the following statements are equivalent.*

(i) *X is a continuous poset.*

(ii) *The counit $\bigvee \colon \mathrm{Idl}(X) \to X$ has a left adjoint; it is $x \mapsto \twoheaddownarrow x$.*

(iii) *X carries a (necessarily unique) Idl_1-coalgebra structure $X \to \mathrm{Idl}(X)$, which is also $\twoheaddownarrow(-)$.*

The equivalence of (i) and (ii) is known from the literature, see e.g. [8, VII, Proposition 2.1] or [6, Proposition 2.3]. The formulation in (iii) gives a new characterization of continuous posets, which is neatly dual to the above characterization of dcpo's.

Proof. We'll do (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii) Define a function $\varphi \colon X \to \mathrm{Idl}(X)$ by $\varphi(x) = \twoheaddownarrow x = \{y \in X \mid y \ll x\}$. We show that φ is an Idl_1-coalgebra, i.e. that it is a continuous function satisfying

$$\bigvee \varphi(x) = x \qquad \text{and} \qquad \downarrow\{\varphi(y) \mid y \in \varphi(x)\} = \downarrow\{\downarrow y \mid y \in \varphi(x)\}$$

Continuity boils down to

$$y \ll \bigvee a \;\Leftrightarrow\; \exists w \in a.\, y \ll w.$$

The implication ($\Leftarrow$) is obvious; for ($\Rightarrow$), assume $y \ll \bigvee a$; then by interpolation (with $f = id$), there is a z with $y \ll z \ll \bigvee a$. But then $z \leq w$ for some $w \in a$ and thus $y \ll w$.

The first of the equations holds by definition. As to the second one:

($\subseteq$) If $z \in \varphi(y)$ for $y \in \varphi(x)$, then $z \ll y$, so $z \leq y$ and $z \in \downarrow y$.

($\supseteq$) If $z \in \downarrow y$ for $y \in \varphi(x)$, then $z \leq y$ and $y \ll x$. By interpolation we get an y' with $y \ll y' \ll x$. Thus $z \ll y'$, and so $z \in \varphi(y')$ where $y' \in \varphi(x)$.

(iii) $\Rightarrow$ (ii) Assume φ is a coalgebra as above. We have to show that for an ideal $a \in \mathrm{Idl}(X)$ and an element $x \in X$ one has

$$\varphi(x) \subseteq a \;\Leftrightarrow\; x \subseteq \bigvee a$$

The implication ($\Rightarrow$) is easy: if $\varphi(x) \subseteq a$, then $x = \bigvee \varphi(x) \leq \bigvee a$. Conversely, assume $x \leq \bigvee a$ and $y \in \varphi(x)$. Then $y \in \varphi(x) \subseteq \varphi(\bigvee a) = \bigcup\{\varphi(z) \mid z \in a\}$. So $y \in \varphi(z)$ for some $z \in a$. But then $y \leq \bigvee \varphi(z) = z \in a$, so $y \in a$.

(Remarkably, the second equation has not been used; so one can weaken the assumption.)

(ii) $\Rightarrow$ (i) We may assume a monotone function $\varphi \colon X \to \mathrm{Idl}(X)$ with $\varphi(x) \subseteq a \;\Leftrightarrow\; x \leq \bigvee a$. We then have $\bigvee \varphi(x) = x$. In one direction, $\varphi(x) \subseteq \varphi(x)$ yields us $x \leq \bigvee \varphi(x)$. And conversely, from $x \leq \bigvee \downarrow x$ we get $\varphi(x) \subseteq \downarrow x$, and so $\bigvee \varphi(x) \leq \bigvee \downarrow x = x$.

It thus suffices to prove $y \in \varphi(x) \;\Leftrightarrow\; y \ll x$.

($\Rightarrow$) If $y \in \varphi(x)$ and $x \leq \bigvee a$. Then $y \in \varphi(x) \subseteq a$.

($\Leftarrow$) If $y \ll x = \bigvee \varphi(x)$, then $y \leq z$ for some $z \in \varphi(x)$. But then $y \in \varphi(x)$, because the latter is a downset. $\square$

There is another interesting aspect: the ideal monad that we started from on the category of posets is a so-called 'KZ-doctrine'; it is a monad for which structure maps (i.e. algebras) are left adjoints to units, see [9]. Here we discovered a further property, namely that costructures (i.e. coalgebras) are left adjoints to structures. We thus see from this result that the sequence of succesive algebras and coalgebras as described in the introduction corresponds to the sequence of succesive left adjoints of the unit $\downarrow(-): X \to \mathrm{Idl}(X)$, as in

$$
\begin{array}{ccccccccc}
\cdots & \leftrightarrows & \mathrm{Alg}(\mathrm{Idl}_2) & \rightleftarrows & \mathrm{CoAlg}(\mathrm{Idl}_1) & \leftrightarrows & \mathrm{Alg}(\mathrm{Idl}_0) & \rightleftarrows & \mathbf{PoSets} \\
 & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
 & & \mathrm{Idl}_3 & & \mathrm{Idl}_2 & & \mathrm{Idl}_1 & & \mathrm{Idl}_0 = \mathrm{Idl} \\
 & & & & & & & & \\
 & & \text{algebra:} & & \text{coalgebra:} & & \text{algebra:} & & \text{unit:} \\
 & & & & & & & & \mathrm{Idl}(X) \\
\cdots & \dashv & \downarrow & \dashv & \uparrow & \dashv & \downarrow & \dashv & \uparrow \ \eta = \downarrow(-) \\
 & & & & & & & & X
\end{array}
$$

All these adjoints live in (the 2-category) **PoSets**.

There is also a characterization of coalgebra maps in terms of $\ll$.

3.3. Proposition. *Assume that X, Y are continuous posets. Coalgebra maps $X \to Y$ (for Idl_1) are then precisely the continuous maps $X \to Y$ which are monotone with respect to the way below relation $\ll$.*

Proof. A map $f: X \to Y$ in **Dcpo** is an Idl_1-coalgebra map if and only if for each $x \in X$,

$$\twoheaddownarrow f(x) = \mathrm{Idl}(f)(\twoheaddownarrow x) = \downarrow\{f(y) \mid y \in \twoheaddownarrow x\} \qquad (*)$$

Assume f is such a coalgebra map. If $y \ll x$, then $f(y)$, being in the set on the right, is also in the set on the left. Thus $f(y) \ll f(x)$, and f is monotone.

In the reverse direction, assume f is monotone with respect to $\ll$. We show that the equation $(*)$ holds.

$$
\begin{array}{rcll}
y \in \twoheaddownarrow f(x) & \Rightarrow & y \ll f(x) & \\
 & \Rightarrow & \exists z.\, y \ll f(z) \;\&\; z \ll x & \text{by interpolation} \\
 & \Rightarrow & \exists z.\, y \leq f(z) \;\&\; z \in \twoheaddownarrow x & \\
 & \Rightarrow & y \in \downarrow\{f(z) \mid z \in \twoheaddownarrow x\} & \\
y \in \downarrow\{f(z) \mid z \in \twoheaddownarrow x\} & \Rightarrow & \exists z.\, y \leq f(z) \;\&\; z \ll x & \\
 & \Rightarrow & \exists z.\, y \leq f(z) \;\&\; f(z) \ll f(x) & \text{since } f \text{ is monotone} \\
 & \Rightarrow & y \ll f(x) & \\
 & \Rightarrow & y \in \twoheaddownarrow f(x). &
\end{array}
$$

$\square$

We conclude that the category of coalgebras $\mathrm{CoAlg}(\mathrm{Idl}_1)$ of the ideal comonad Idl_1 on **Dcpo** has a nice description as the category of continuous posets and functions which are Scott-continuous and monotone with respect to the way below relation. This category has finite products, but it does not seem to be cartesian closed. Hence it is not so interesting as a model for logic.

Aside: it is not clear how these coalgebra maps relate to 'Lawson maps' between continuous posets, see [8, VII, 2.10].

But one can go a step further and try to characterize the next step in the sequence—corresponding again to an algebra, namely of the monad Idl_2, or to a left adjoint to $\twoheaddownarrow(-)$. The equivalence between (i) and (ii) in the next result comes from [6] (see also [11]), but the link with algebras is new again.

3.4. Theorem. *For a continuous poset X, the following are equivalent.*

(i) *Every directed subset $a \subseteq X$ has a $\leq$-least $\ll$-upper bound, written as $\underline{\sup}(a)$.*

(ii) *The function $\twoheaddownarrow(-): X \to \mathrm{Idl}(X)$ has a left adjoint.*

(iii) *The continuous poset X carries a (necessarily unique) Idl_2-algebra structure $\mathrm{Idl}(X) \to X$.*

Proof. We'll do (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) Define a function $\varphi: X \to \mathrm{Idl}(X)$ by $\varphi(a) = \underline{\sup}(a)$. We have to show that φ is continuous and monotone (wrt. $\ll$) and satisfies the algebra equations,

$$\varphi(\twoheaddownarrow x) = x \qquad \text{and} \qquad \varphi(\downarrow\{\bigvee a \mid a \in A\}) = \varphi(\downarrow\{\varphi(a) \mid a \in A\}).$$

For the first of these we notice that x is a $\ll$-upperbound of $\twoheaddownarrow x$, so $\underline{\sup}(\twoheaddownarrow x) \leq x$. Conversely, we have for $y \in \twoheaddownarrow x$ that $y \ll \underline{\sup}(\twoheaddownarrow x)$ and so $y \leq \underline{\sup}(\downarrow x)$. Then $x = \bigvee \twoheaddownarrow x \leq \underline{\sup}(\twoheaddownarrow x)$.

The second equation is broken up:

($\leq$) Assume $x \leq \bigvee a_0$ for some $a_0 \in A \subseteq \mathrm{Idl}(X)$. Then $\twoheaddownarrow x \subseteq a_0$ and so $x = \underline{\sup}(\twoheaddownarrow x) \leq \underline{\sup}(a_0)$. We need to show $x \ll \underline{\sup}(\downarrow\{\underline{\sup}(a) \mid a \in A\})$. But this follows from $x \in \downarrow\{\underline{\sup}(a) \mid a \in A\}$.

($\geq$) Assume $x \leq \underline{\sup}(a_0)$ for $a_0 \in A$. We must show $x \ll \underline{\sup}(\downarrow\{\bigvee a \mid a \in A\})$. So suppose $\underline{\sup}(\downarrow\{\bigvee a \mid a \in A\}) \leq \bigvee_i z_i$. Then $\downarrow\{\bigvee a \mid a \in A\} \subseteq \twoheaddownarrow \bigvee_i z_i = \bigcup_i \twoheaddownarrow z_i$. Hence $\bigvee a_0 \in \twoheaddownarrow z_i$ for some i, and so $a_0 \subseteq \twoheaddownarrow z_i$. Hence $x \leq \underline{\sup}(a_0) \leq z_i$.

Next we have to show that $a \mapsto \underline{\sup}(a)$ is continuous and monotone with respect to $\ll$. It is obvious that $\underline{\sup}$ is monotone with respect to $\subseteq$, so continuity boils down to $\underline{\sup}(\bigcup_i a_i) \leq \bigvee_i \underline{\sup}(a_i)$ for a directed collection of $a_i \in \mathrm{Idl}(X)$. If $x \in \bigcup_i a_i$, then $x \in a_{i_0}$ for some i_0. Then $x \ll \underline{\sup}(a_{i_0}) \leq \bigvee_i \underline{\sup}(a_i)$. Hence $\bigvee_i \underline{\sup}(a_i)$ is a $\ll$-upper bound for $\bigcup_i a_i$, which yields the required result.

If $a \ll b$ in $\mathrm{Idl}(X)$, then $\underline{\sup}(a) \ll \underline{\sup}(b)$ in X. Indeed, if $\underline{\sup}(b) \leq \bigvee_i x_i$, then for $y \in b$ we have $y \ll \underline{\sup}(b) \leq \bigvee_i x_i$. Hence $y \ll x_i$ for some i. This yields $b \subseteq \bigcup_i \twoheaddownarrow x_i$. But then $a \subseteq \twoheaddownarrow x_i$ for some i, and thus $\underline{\sup}(a) \leq x_i$.

(iii) $\Rightarrow$ (ii) Let φ be an algebra, as above. We have to prove $\varphi(a) \leq x \;\Leftrightarrow\; a \subseteq \twoheaddownarrow x$. The implication ($\Leftarrow$) is immediate: if $a \subseteq \twoheaddownarrow x$, then $\varphi(a) \leq \varphi(\twoheaddownarrow x) = x$. Conversely, assume $\varphi(a) \leq x$ and $y \in a$. In order to get $y \ll x$, it suffices to have $\twoheaddownarrow y \ll a$ in $\mathrm{Idl}(X)$, because using monotony of φ, we then get $y = \varphi(\twoheaddownarrow y) \ll \varphi(a) \leq x$. Thus we assume $a \subseteq \bigcup_i b_i$. Since $y \in a$, we get $y \in b_i$ for some i. But then $\twoheaddownarrow y \subseteq \downarrow y \subseteq b_i$.

(ii) $\Rightarrow$ (i) Assume $\varphi\colon \mathrm{Idl}(X) \to X$ is monotone and satisfies $\varphi(a) \leq x \;\Leftrightarrow\; a \subseteq \twoheaddownarrow x$. We wish to show that for a directed subset $b \subseteq X$, one has that $\varphi(\downarrow b) = \underline{\sup}(b)$, i.e. that $\varphi(\downarrow b)$ is $\leq$-least $\ll$-upperbound of b.

From $\downarrow b \subseteq \twoheaddownarrow \varphi(\downarrow b)$ we get $x \in b \;\Rightarrow\; x \ll \varphi(\downarrow b)$. Thus $\varphi(\downarrow b)$ is an $\ll$-upperbound of b. If y is another such, then $\downarrow b \subseteq \twoheaddownarrow y$, so $\varphi(\downarrow b) \leq y$. $\square$

3.5. Lemma. *The Idl_2-algebra maps $f\colon X \to Y$ are the continuous functions which are monotone with respect to $\ll$ and preserve $\underline{\sup}$.*

Proof. And Idl_2-algebra map $f\colon X \to Y$ is an Idl_1-coalgebra map satisfying $f \circ \underline{\sup}_X = \underline{\sup}_Y \circ \mathrm{Idl}(f)$, i.e. $f(\underline{\sup}_X(a)) = \underline{\sup}_Y(\{f(x) \mid x \in a\})$. $\square$

Hence we know what the category $\mathrm{Alg}(\mathrm{Idl}_2)$ of Idl_2-algebras is. It turns out to be equivalent to the category $\mathrm{Alg}(\mathrm{Idl}_0) = \mathbf{Dcpo}$ occurring two steps earlier in the unfolding of Idl on **PoSets**. Therefore we need the following lemma. Most of it is extracted from [6, 11].

3.6. Lemma. *Consider a continuous poset X which carries an Idl_2-algebra $\underline{\sup}$. For an element $x \in X$ write $c(x) = \underline{\sup}(\downarrow x)$. Then*

(i) *The function $c\colon X \to X$ is a closure operation and its closed elements (x with $c(x) = x$) are precisely the compact elements (x with $x \ll x$).*

(ii) *The set $\mathcal{K}(X)$ of compact elements of X is a dcpo.*

(iii) *The poset X is algebraic, and so $X \cong \mathrm{Idl}(\mathcal{K}(X))$.*

(iv) *Maps $f\colon X \to Y$ of Idl_2-algebras preserve compact elements, and are thus determined by their restrictions $\mathcal{K}(X) \to \mathcal{K}(Y)$. These restrictions are continuous functions.*

Proof. (i) If $x \leq y$, then $\downarrow x \subseteq \downarrow y$ and so $c(x) = \underline{\sup}(\downarrow x) \leq \underline{\sup}(\downarrow y) = c(y)$. Hence c is monotone. Further, from $\twoheaddownarrow x \subseteq \downarrow x$ we get $x = \underline{\sup}(\twoheaddownarrow x) \leq \underline{\sup}(\downarrow x) = c(x)$. And $c^2(x) \leq c(x)$ holds because $\downarrow \underline{\sup}(\downarrow x) \subseteq \twoheaddownarrow \underline{\sup}(\downarrow x)$. Indeed, if $y \leq \underline{\sup}(\downarrow x)$, then $y \ll \underline{\sup}(\downarrow x)$: if $\underline{\sup}(\downarrow x) \leq \bigvee_i z_i$, then $\downarrow x \subseteq \twoheaddownarrow \bigvee_i z_i = \bigcup_i \twoheaddownarrow z_i$, so that $x \in \twoheaddownarrow z_i$ for some i, and hence $\downarrow x \subseteq \twoheaddownarrow z_i$. Hence $y \leq \underline{\sup}(\downarrow x) \leq \underline{\sup}(\twoheaddownarrow z_i) \leq z_i$.

Next we show that x is compact if and only if $c(x) = x$. If x is compact, then $x \ll x$ and so $\downarrow x \subseteq \twoheaddownarrow x$. Hence $c(x) = \underline{\sup}(\downarrow x) \leq \underline{\sup}(\twoheaddownarrow x) = x$ and thus $c(x) = x$. Conversely, if $c(x) = x$, then $\underline{\sup}(\downarrow x) \leq x$ and so $\downarrow x \subseteq \twoheaddownarrow x$. This yields that $x \ll x$.

(ii) As join of a directed set $a \subseteq \mathcal{K}(X)$ one takes $\bigsqcup a = c(\bigvee a) \in \mathcal{K}(X)$, where $\bigvee a$ is the join in X.

(iii) For $x \in X$, the set $\{z \in \mathcal{K}(X) \mid z \leq x\}$ of compact elements below x is directed: if $z_1, z_2 \in \mathcal{K}(X)$ satisfy $z_1, z_2 \leq x$, then $z_1, z_2 \ll x$, because z_1, z_2 are compact. So there is an $y \ll x$ with $z_1, z_2 \leq y$, since X is a continuous poset. Then $\downarrow y \subseteq \twoheaddownarrow x$, and so $c(y) = \underline{\sup}(\downarrow y) \leq x$. Hence we have $c(y) \in \mathcal{K}(X)$ with $z_1, z_2 \leq c(y)$ and $c(y) \leq x$.

Since $y \ll x \;\Rightarrow\; y \leq c(y) \leq x$ (as just shown), we get

$$\{y \in X \mid y \ll x\} \subseteq \downarrow\{c(y) \mid y \in X \;\&\; c(y) \leq x\} = \downarrow\{z \in \mathcal{K}(X) \mid z \leq x\}$$

and so $x = \bigvee\{y \in X \mid y \ll x\} \leq \bigvee\{z \in \mathcal{K}(X) \mid z \leq x\}$. The reverse $(\geq)$ holds trivially.

(iv) If $f\colon X \to Y$ is an Idl_2-algebra map, then it preserves $\underline{\sup}$ by the previous lemma. Hence

$$f(c(x)) = f(\underline{\sup}(\downarrow x)) = \underline{\sup}\downarrow\{f(y) \mid y \leq x\} = \underline{\sup}(\downarrow f(x)) = c(f(x)).$$

Thus if x is compact, then $f(x) = f(c(x)) = c(f(x))$ so $f(x)$ is compact. So $f\colon X \to Y$ restricts to $f\colon \mathcal{K}(X) \to \mathcal{K}(Y)$. This restriction is continuous, since for directed $a \subseteq \mathcal{K}(X)$ one has

$$f(\bigsqcup a) = f(c(\bigvee a)) = c(f(\bigvee a)) = c(\bigvee\{f(x) \mid x \in a\}) = \bigsqcup\{f(x) \mid x \in a\}.$$

Since each $x \in X$ is join of compacts below x, the function $f\colon X \to Y$ is determined by its behaviour on compact elements. $\square$

3.7. Corollary. *The compact element functor*

$$\mathcal{K}\colon \mathrm{Alg}(\mathrm{Idl}_2) \longrightarrow \mathrm{Alg}(\mathrm{Idl}_0) = \mathbf{Dcpo}$$

is an equivalence. $\square$

Hence the sequence starting from the ideal monad on **PoSets** reaches a fixed point in three unfoldings.

4 Coalgebras and categories with coproducts

There is an interesting aspect about the second example, which forms the basis for the third. As already mentioned, the ideal monad Idl on the category of posets has the special property that structure maps (i.e. algebras) are left adjoints to units; this is precisely the content of the equivalance of (ii) and (iii) in Proposition 3.1. Monads with this property are studied in [9]; it can be shown that the resulting correspondence between costructures (i.e. coalgebras) and left adjoints of the counits is a more general phenonmenon for such monads. This suggests to consider other such monads, for example the family monad Fam on the category **Cat** of categories and functors.

For an arbitrary category $\mathbb{C}$ one writes $\mathrm{Fam}(\mathbb{C})$ for the associated category of set indexed families of objects and morphisms in $\mathbb{C}$. An object in $\mathrm{Fam}(\mathbb{C})$ is a collection $(X_i)_{i\in I}$ of objects $X_i \in \mathbb{C}$. A morphism $(X_i)_{i\in I} \to (Y_j)_{j\in J}$ is a pair (u, f) where $u: I \to J$ is a function between the index sets and $f = (f_i)_{i\in I}$ is a collection of morphisms $f_i: X_i \to Y_{u(i)}$ in $\mathbb{C}$. This $\mathrm{Fam}(-)$ construction is functorial: for a functor $F: \mathbb{C} \to \mathbb{D}$ one has $\mathrm{Fam}(F): \mathrm{Fam}(\mathbb{C}) \to \mathrm{Fam}(\mathbb{D})$ given by $(X_i)_{i\in I} \mapsto (FX_i)_{i\in I}$. We get that Fam is a (pseudo) monad on **Cat**. Actually, there are some isomorphisms involved, which we ignore in order not to complicate matters. The unit $\eta: \mathbb{C} \to \mathrm{Fam}(\mathbb{C})$ is given by indexing over a terminal, one-element set 1. Algebras for this monad are categories with coproducts, i.e. categories in which each set indexed collection $(X_i)_{i\in I}$ of objects has a coproduct $\coprod_{i\in I} X_i$. Algebra maps are coproduct preserving functors. And indeed $\mathbb{C}$ has coproducts if and only if the unit $\eta: \mathbb{C} \to \mathrm{Fam}(\mathbb{C})$ has a left adjoint $\coprod$. This is a result as in Proposition 3.1 and shows that $\mathrm{Fam}(-)$ is like $\mathrm{Idl}(-)$.

An object X in a category $\mathbb{C}$ with coproducts is called **indecomposable** if for each collection $(Y_i)_{i\in I}$ one has

$$X \cong \coprod_{i\in I} Y_i \quad \Rightarrow \quad \exists i \in I.\, X \cong Y_i$$

Notice that an indecomposable object is not initial. We write $\mathcal{I}(\mathbb{C})$ for the collection of indecomposable objects in $\mathbb{C}$. One calls a category $\mathbb{C}$ a **atomic** if each object can be written as coproduct of indecomposables. For example, **Sets** is atomic with singletons as indecomposable objects.

Most of the ingredients of the proof of the next result are known, but the connection that is established (especially with coalgebras) is new. It came up in discussion with Anders Kock. The result is an analogue of Theorem 3.2.

4.1. Theorem. *Let $\mathbb{C}$ be a category with finite limits and (chosen) universal and disjoint coproducts. Then the following statements are equivalent.*

(i) *The coproduct functor $\coprod: \mathrm{Fam}(\mathbb{C}) \to \mathbb{C}$ has a left adjoint $\mathcal{D}$.*
[Since $\eta: \mathbb{C} \to \mathrm{Fam}(\mathbb{C})$ is full and faithful, the unit of the adjunction $(\mathcal{D} \dashv \coprod)$ is necessarily an isomorphism. Equivalently, $\mathcal{D}$ is full and faithful.]

(ii) The category $\mathbb{C}$ is atomic.

(iii) The coproduct functor $\coprod: \mathrm{Fam}(\mathcal{I}(\mathbb{C})) \to \mathbb{C}$ is an equivalence.

(iv) The category $\mathbb{C}$ carries a coalgebra structure $\mathcal{D}: \mathbb{C} \to \mathrm{Fam}(\mathbb{C})$. □

Proof. (i) ⇒ (ii) Let's fix an object $X \in \mathbb{C}$ and write $\mathcal{D}(X) = (X_i)_{i\in I}$. Then $X = \coprod_{i\in I} X_i$. Our aim is to show that each X_i is indecomposable. First we establish: X_i is not initial. Else put $I' = I - \{i\}$; there are then many morphisms $(X_i)_{i\in I} \to (X_i)_{i\in I'}$ which give rise to the map $X = \coprod_{i\in I} X_i \xrightarrow{\sim} \coprod_{i\in I'} X_i$. This is in contradiction with the adjointness.

Next assume an isomorphism $\varphi\colon X_i \xrightarrow{\sim} \coprod_{j\in J} Y_j$ with $J \neq \emptyset$. Write K for the disjoint union $(I - \{i\}) + J$ and define Z_k to be X_k if $k \in I - \{i\}$ and Y_k if $k \in J$. We get an iso $\psi\colon X \xrightarrow{\sim} \coprod_{k\in K} Z_k$, so by the adjunction there is a unique morphism $(u, f)\colon (X_i)_{i\in I} \to (Z_k)_{k\in K}$ for which the composite

$$X = \coprod_{i\in I} X_i \xrightarrow{\coprod(u,f)} \coprod_{k\in K} Z_k \qquad \text{is} \qquad X \xrightarrow{\psi} \coprod_{k\in K} Z_k$$

One distinguishes (a) $u(i) \in I - \{i\}$ and (b) $u(i) \in J$. In the first case can form the pullback,

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d, "X_{u(i)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & & X_i \arrow[d, "\mathsf{in}_i"] \\ X_{u(i)} \arrow[rr, "\mathsf{in}_{u(i)}"'] & & \coprod_{i\in I} X_i \end{tikzcd}$$

This yields a map $X_i \xrightarrow{\sim} 0$, which is impossible.

In the second case, write $j = u(i) \in J$, then the composite

$$X_i \xrightarrow{f_i} Z_j \overset{\mathsf{in}_j}{\rightarrowtail} \coprod_{k\in K} Z_k \qquad \text{is} \qquad X_i \underset{\sim}{\xrightarrow{\varphi}} \coprod_{k\in K} Z_k$$

This yields that f_i is an isomorphism $X_i \cong Z_j = Y_j$.

(ii) $\Leftrightarrow$ (iii) Assume (ii). We choose a functor $\mathcal{D}\colon \mathbb{C} \to \mathrm{Fam}(\mathcal{I}(\mathbb{C}))$ by $\mathcal{D}(X) = (X_i)_{i\in I}$ where $\{X_i \mid i \in I\}$ is the set of indecomposables with X as coproduct. Thus $\coprod \mathcal{D}(X) = X$. Such a set can be chosen only up-to-isomorphism in $\mathrm{Fam}(\mathbb{C})$. If $\coprod_{i\in I} X_i = X = \coprod_{j\in J} Y_j$ with all X_i and Y_j indecomposable, then consider the pullback,

$$\begin{tikzcd} Z_{i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_j \arrow[d, "\mathsf{in}_j"] \\ X_i \arrow[r, "\mathsf{in}_i"'] & X \end{tikzcd}$$

By universality we get $X_i \cong \coprod_{j\in J} Z_{i,j}$ and $Y_j \cong \coprod_{i\in I} Z_{i,j}$. Hence for each $i \in I$ there is a $j_i \in J$ with $X_i \cong Z_{i,j_i}$ and for each $j \in J$ there is a $i_j \in I$ with $Y_j \cong Z_{i_j,j}$. One can prove that $i = i_{j_i}$ and $j = j_{i_j}$. It is left to the reader to define $\mathcal{D}$ on morphisms.

The implication (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iv) The above functor $\mathcal{D}$—with $\mathcal{D}X = (X_i)_{i\in I}$, where X_i indecomposable—extends to a coproduct preserving functor $\mathbb{C} \to \mathrm{Fam}(\mathbb{C})$. It is a coalgebra since

$$(\varepsilon \circ \mathcal{D})(X) = \coprod_{i\in I} X_i = X$$

and

$$\begin{aligned}(\mathrm{Fam}(\mathcal{D}) \circ \mathcal{D})(X) \;=\; \big(\mathcal{D}(X_i)\big)_{i\in I} &= \big(\eta(X_i)\big)_{i\in I} \\ &= (\mathrm{Fam}(\eta) \circ \mathcal{D})(X) \\ &= (\delta \circ \mathcal{D})(X).\end{aligned}$$

where $\mathcal{D}(X_i) = \eta(X_i)$ holds since X_i is indecomposable. It is not hard to verify that $\mathcal{D}$ preserves coproducts.

(iv) $\Rightarrow$ (i) Assume we have a coalgebra $\mathcal{D}\colon \mathbb{C} \to \mathrm{Fam}(\mathbb{C})$, which thus satisfies the above equations, and preserves coproducts. For a (fixed) object $X \in \mathbb{C}$, write $\mathcal{D}(X) = (X_i)_{i\in I}$. Then $\coprod_{i\in I} X_i = X$ and $\mathcal{D}(X_i) = \eta(X_i)$. We claim that X_i is indecomposable: if $X_i \cong \coprod_{j\in J} Y_j$, then

$$\eta(X_i) \;=\; \mathcal{D}(X_i) \;\cong\; \mathcal{D}(\coprod_{j\in J} Y_j) \;\cong\; \coprod_{j\in J} \mathcal{D}(Y_j)$$

Since $\eta(X_i)$ is indecomposable in $\mathrm{Fam}(\mathbb{C})$, we get $\eta(X_i) \cong \mathcal{D}(Y_j)$ for some $j \in J$. Then $X_i = \coprod \mathcal{D}(X_i) = \coprod \eta(X_i) \cong \coprod \mathcal{D}(Y_j) = Y_j$.

For the adjunction $(\mathcal{D} \dashv \coprod)$ we have to establish a bijective correspondence

$$\mathcal{D}(X) = \frac{(X_i)_{i\in I} \;\longrightarrow\; (Z_k)_{k\in K}}{X \;\longrightarrow\; \coprod_{k\in K} Z_k}$$

Downwards, one takes the coproduct $\coprod$. Upwards, assume $v\colon X \to \coprod_{k\in K} Z_k$, and form the pullback diagrams,

$$\begin{tikzcd} W_{(i,k)} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & Z_k \arrow[d, "\mathsf{in}_k"] \\ X_i \arrow[r, "\mathsf{in}_i"] & X \arrow[r, "v"] & \coprod_{k\in K} Z_k \end{tikzcd}$$

By universality, we get $X_i \cong \coprod_k W_{(i,k)}$. This yields an $u(i) \in K$ with $X_i \cong W_{(i,u(i))}$, and thus a map $X_i \cong W_{(i,u(i))} \to Z_k$. $\square$

In going from $\mathbb{C}$ to $\mathrm{Fam}(\mathbb{C})$ one freely adds arbitrary (set-indexed) coproducts to $\mathbb{C}$. There are similar completions like

$\mathbb{C}$	$\mapsto$	**Ind**-$\mathbb{C}$	adding (small) filtered colimits (see [8, VI])
$\mathbb{C}$	$\mapsto$	$\mathbf{Sets}^{\mathbb{C}^{op}}$	adding all (small) colimits

which can be analyzed in the same way. We expect similar approximation results, characterizing coalgebras as those categories in which all objects are colimits of suitable atoms. With more examples like these the need becomes more urgent to put matters in a proper 2-categorical framework.

Acknowledgement. I should thank Anders Kock for stimulation discussions.

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