Quantum Processes and Computation

Assignment 4, Wednesday, February 17, 2019

Exercise teachers:
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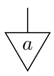
Handing in your answers: There are two options:

1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to `wetering@cs.ru.nl`. Please include your name and the exercise number in the filename, e.g. `ACHTERNAAM-qpc-exercise4.pdf`.

Deadline: Tuesday, February 26, 12:00

Goals: After completing these exercises you know about orthonormal bases, composition of linear maps and logic gates as linear maps. The total number of points is 100, distributed over 5 exercises. Material covered in book: Sections 5.1, 5.2, 5.3.4 and a bit of 5.3.5

Exercise 1 (5.4) (20 points): We saw in the lecture that for a set A with n elements in **relations** the singletons:

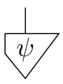
$$\mathcal{B}_A := \left\{ \quad \right.$$


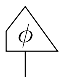
$$\left. \middle| \; a \in A \right\}$$

form a basis, that is, that no element can be removed from $\mathcal{B}_A$ without losing the property of being a basis. This basis is also orthonormal. Show that this is the *only* orthonormal basis of A.

Bonus exercise: The orthonormality condition is actually not necessary for proving the uniqueness of the basis. Show that any basis (not necessarily orthonormal) of A must be the singleton basis.

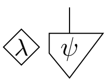
Exercise 2 (5.54) (20 points): Let



$$\leftrightarrow \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} \qquad \text{and}$$


$$\leftrightarrow \begin{pmatrix} \phi_0 & \phi_1 \end{pmatrix}$$

be respectively a 2-dimensional state, and 2-dimensional effect. Let λ be a number. Write the matrices for the processes

(*i*)


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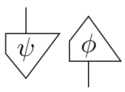

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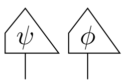
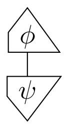


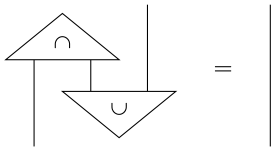
(*iv*)


Exercise 3 (5.58) (20 points): The matrices for cups and caps in 2 dimensions are:


$$\leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

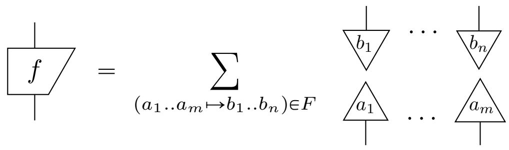
(i) First, verify the yanking equation



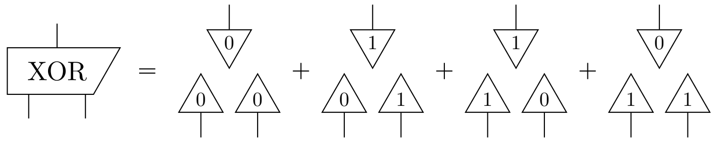
directly using the matrices of the 2-dimensional cup and cup by using the rules for sequential and parallel composition of matrices, i.e. show that $(\cap \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \cup) = \mathrm{id}$ (where id is the 2×2 identity matrix).

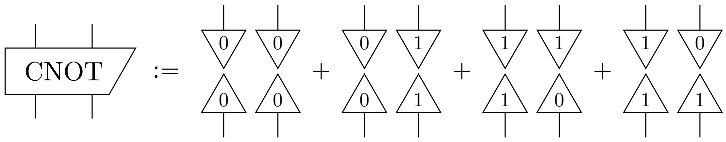
(ii) Second, give the matrices for the cup and cap in 3 dimensions.

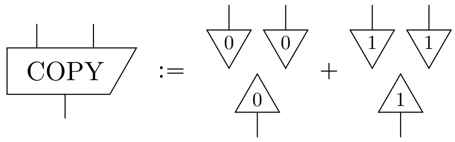
The next two excercises are about encoding classical logic gates in the theory of **linear maps**, as explained in Section 5.3.4. Recall that a classical logic gate F can be encoded as a linear map via:



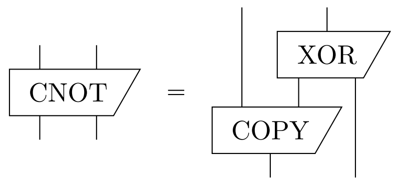
Using this encoding, we defined:





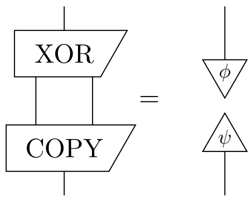


Exercise 4 (5.86) (20 points): Show that



(Hint: try comparing the LHS to the RHS on all basis states, rather than writing out a big sum.)

Next, find ψ and ϕ such that the following equation holds:



Although it might not look like much now, this equation will turn out to lie at the heart of the notion of *complementarity* which we will cover in great depth in the coming lectures.

Exercise 5 (based on 5.83) (20 points): Show that if a logic gate has an inverse, its associated linear map is unitary.