Quantum Processes and Computation

Assignment 7, Wednesday, March 13, 2019

Exercise teachers:
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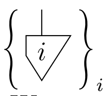
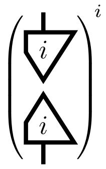
Handing in your answers: There are two options:

1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to `wetering@cs.ru.nl`. Please include your name and the exercise number in the filename, e.g. `ACHTERNAAM-qpc-exercise1.pdf`.

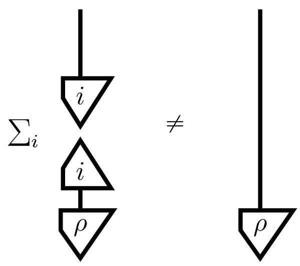
Deadline: Tuesday, April 9, 12:00

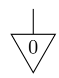
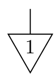
Goals: After completing these exercises you will know how to calculate the probabilities of measurement outcomes and how different measurements combine together. The total number of points is 100, distributed over 3 exercises.
Material covered in book: sections 6.4, 7.1 and 7.2.

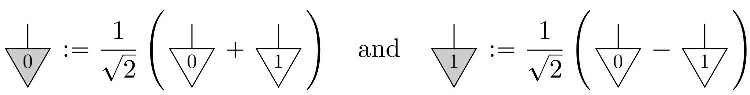
Exercise 1 (7.13) (20 points): Let  form an ONB for any non-trivial system (so that it contains at least two elements). We can use this ONB to form a non-demolition ONB measurement in the following way:



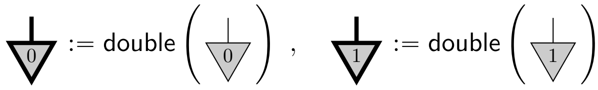
If we sum over all the branches we get a process that is called *decoherence*. Show that decoherence is not equal to the identity by finding an explicit quantum state ρ such that:

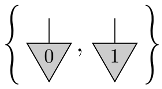


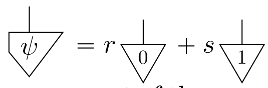
Exercise 2 (50 points): In this exercise our system is a qubit. Let  and  denote the standard basis states for $\mathbb{C}^2$, and let



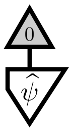
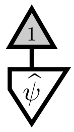
denote the *X-basis*. Note that, like the standard basis states, these are self-conjugate, so we write them as symmetric triangles. The associated quantum states are written as:

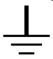


(i) Show explicitly that  forms an ONB for $\mathbb{C}^2$.

(ii) Let $r, s \in \mathbb{R}$ be such that $r^2 + s^2 = 1$ so that the state  is normalised.

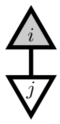
Show that the probabilities associated to the X-basis measurement of the pure quantum state $\widehat{\psi}$ in terms of r and s are

 $= \frac{1}{2} + rs$ and  $= \frac{1}{2} - rs$

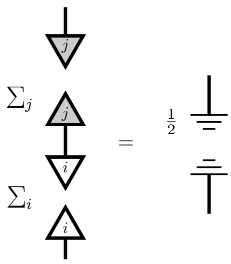
(iii) Show that the two possible outcomes of an ONB measurement when applied to the maximally mixed state $\frac{1}{2}$  have equal probability (regardless of the chosen ONB).

(iv) By the previous point, when we measure the maximally mixed state in the X-basis, both outcomes are equally likely. Show that however, if we do a non-demolition measurement in the X-basis *and then* we do another (demolition) measurement in the X-basis, that we will never get a different outcome than the one we've seen in the first (non-demolition) measurement. So if the outcome was 0 the first time, it will never be 1 the second time around.

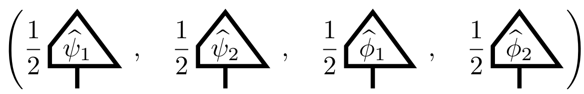
(v) First show that for all $i, j \in \{0, 1\}$

 $= \frac{1}{2}$

and use this to show that if we apply the decoherence channel of the standard basis, and then we apply the decoherence channel of the X-basis, that we get the *completely decohering* channel (also known as the *noise channel*):



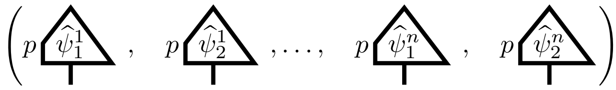
Exercise 3 (30 points): Suppose we have two different ONB measurements on a qubit. Before we do anything with the quantum system we can flip a coin. If it comes up heads we can perform the first ONB measurement, and if it comes up tails we can do the other one. This is modelled by the quantum process



where $(\widehat{\psi}_1, \widehat{\psi}_2)$ and $(\widehat{\phi}_1, \widehat{\phi}_2)$ both form ONB measurements.

(i) Show that the above set of 4 effects indeed forms a quantum measurement (i.e. that it satisfies the causality condition).

(ii) We can generalise the above construction. Suppose we now have n different ONB measurements which we call $(\widehat{\psi}_1^j, \widehat{\psi}_2^j)$ for $j = 1, \ldots, n$. Find the number p such that



is a quantum measurement. Would the number p change if instead of ONB measurements on a qubit, we would consider ONB measurements on a bigger system?