Quantum Processes and Computation

Midterm Wednesday, April 24, 2019

Exercise teachers:
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Handing in your answers: There are two options:

1. Deliver a hard copy to the mailbox of John van de Wetering. Mercator 1, 3rd floor.
2. E-mail a PDF to `wetering@cs.ru.nl`. Please include your name and the exercise number in the filename, e.g. `ACHTERNAAM-qpc-exercise1.pdf`.

Deadline: Wednesday, May 15, 17:00

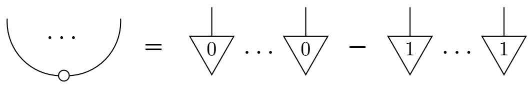
Goals: The goal of this assignment is to assess a student's abilities to do diagrammatic reasoning. Namely, students must use a previously unencountered set of graphical equations to perform calculations and prove theorems. The total number of points is 150, distributed over 9 exercises.

In the main course we have slowly been introducing new graphical features which we have used to study quantum theory. This culminated in the presentation of the ZX-calculus as a complete description of all quantum processes. This is however not the only graphical calculus that can be used to describe quantum processes, and depending on what you want to use it for it might be more useful to use some other graphical calculus. That is exactly what we will be doing in this midterm.

We will work with a subset of the tools we have done before. For the remainder of this assignment we will assume to be working on an undoubled two dimensional system (so $\mathbb{C}^2$), and furthermore everything is self-conjugate, so that there are in principle no complex numbers (and the adjoint is just the transpose).

In the ZX-calculus everything is done with two different colours of spiders that are strongly complementary. Here we will also use two different spiderlike maps, that have quite a different relation.

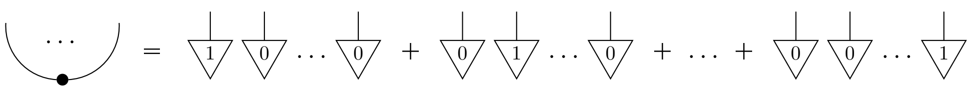
We define the *white spiderlike map* as



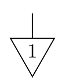
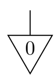
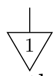
(1)

(note that this is very similar to the regular white spider except for the minus sign). We use the small white dot to distinguish it from the regular white spiders we have seen before. We will not be using those spiders in this assignment.

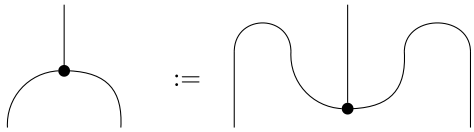
And we define the *black* spiderlike map as



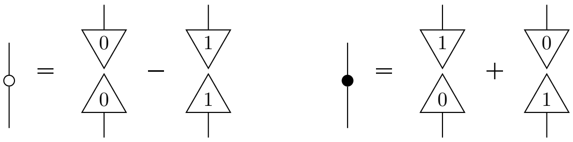
(2)

i.e. every term contains exactly one  and all the other states are . In particular, the black spiderlike map with a single output  is just .

By the symmetry in their definitions we see that these things remain the same if we swap some legs, we can therefore without ambiguity also define a 'spider' with n input legs and m outputs by starting with a $n + m$ wired map and bending n legs down. For instance:



In this way we can construct some of the gates we have seen before:

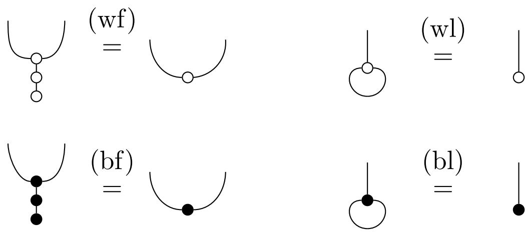


We refer to these maps as spiderlike because they don't act exactly like the spiders we have encountered before.

1 The axioms

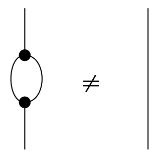
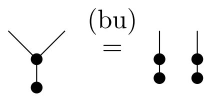
Exercise 1 (20 points):

(i) Demonstrate the spideryness of the maps by using definitions (1) and (2) to show that



(3)

(ii) Demonstrate the non-spideryness of the black-dot maps by proving the following (in)equalities using definition (2):

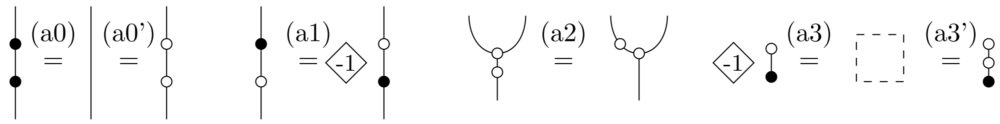
(a)  (b) 

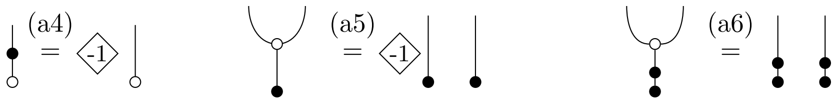
For the remainder of this section, we will provide some additional equations that you may use later. **You do not need to prove these.** However, in the calculations to follow, you should use the labels provided to indicate which equation you are using. If you wish to prove intermediate equations (recommended), you should give these unique labels as well.

The equations of (3) can be generalised to the following spider equations:

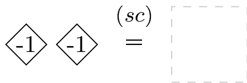
Because of how we have defined these spiderlike maps, we can freely bend legs up and down. So, the equations above can equivalently be presented as:

Some other equations between the spiderlike maps also hold:





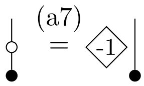
...and of course:



From this point onwards you may not use definitions (1) and (2). The only things you can use are the labeled equations (and variations on them where legs are bent up or down)

Note that the equations we have seen so far are not symmetric in colour. In particular, (a5) has no colourchanged counterpart.

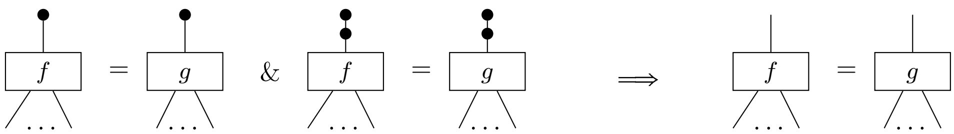
Exercise 2 (10 points): Use the axioms (a0-6) and (wf) to show that (a4) does have a colour changed counterpart:



By cheating and looking at definition (2) we know that $|0\rangle =$  and $|1\rangle =$ . But in fact, using just the rules above we can also derive that these states must be linearly independent.

Exercise 3 (10 points): Show that  is linearly independent of  by deriving a contradiction from the assumption that  $\approx$ . Hint: show that the identity separates. You may use all the equations of this section, but **do not use definitions (1) and (2)**.

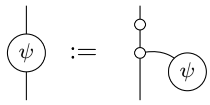
Since  and  are linearly independent and we are working on a two-dimensional space, they form a basis. This means we can prove the equality of two diagrams by plugging in  and  and seeing whether the resulting expressions are equal:

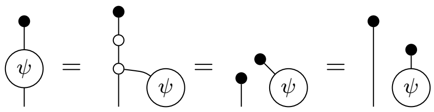


We will refer to this as **proof by plugging**. Since we are free to bend wires in any way, it of course doesn't matter if we plug an output or an input. **Note the equations to the left of "$\Longrightarrow$" must be on-the-nose, not just $\approx$.**

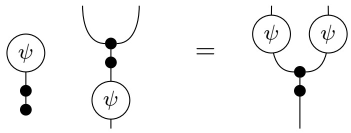
Exercise 4 (30 points): Use proof by plugging to show that white copies through black and vice-versa:

2 Multiplier maps

We define a *multiplier* to be a map of the form: 

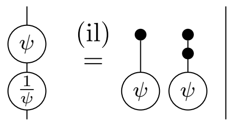
Note that: 

Exercise 5 (20 points): Use proof by plugging to show that multipliers "copy through" black dots:

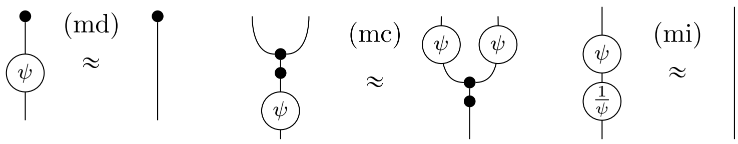


We can also define the *inverse multiplier* as:

Exercise 6 (10 points): Use proof by plugging to show that:



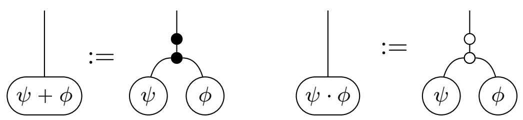
In these last few equations we were left with a combination of scalars $\psi \circ$ or $\psi \circ$. If we assume that these are nonzero and we only care about equality up to a scalar than the equations can be brought into a much nicer form:



(4)

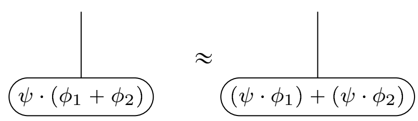
3 Arithmetic

In this section we will assume that the states ψ we will be working with are such that $\psi \circ$ and $\psi \circ$ are nonzero so that the equations in (4) hold. We can then define both *addition* and *multiplication* as follows:

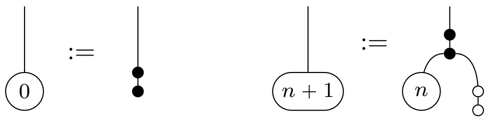


These operations are obviously commutative, and using equations (ws) and (bs) we can also show that they are associative.

Exercise 7 (10 points): Prove that multiplication distributes over addition:



We can now start to define the natural numbers inductively:

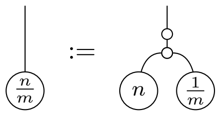


Exercise 8 (20 points):

(i) Using the rewrite rules, show that the state corresponding to the natural number 1 indeed acts as the unit for multiplication.

(ii) Show that for a state ψ we indeed have $\psi \cdot \frac{1}{\psi} \approx 1$ (where 1 is the state corresponding to the natural number 1).

Since we have multiplicative inverses and natural numbers we can in fact create all positive rational numbers:

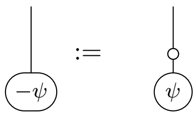


Exercise 9 (20 points):

(i) Show that $\frac{1}{\psi \cdot \phi} = \frac{1}{\psi} \cdot \frac{1}{\phi}$.

(ii) Show that our definition of rational number is well defined by establishing that $\frac{k \cdot n}{k \cdot m} \approx \frac{n}{m}$ (where n, m, k are all states corresponding to natural numbers, as defined above).

This is where the exercises end, but for completeness sake we should note that we can in fact embed *all* the rational numbers, including the negative ones. This is done by defining 'minus' via



and then checking that indeed $\psi + (-\psi) = 0$.