DIAGONAL ARGUMENTS AND LAWVERE'S THEOREM

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Abstract. Overview of the Lawvere's fixed point theorem and some of its applications.

Category theory

Categories. A *category* $\mathcal{C}$ is a collection of *objects* $\mathcal{C}_0$ and *arrows* $\mathcal{C}_1$, such that each arrow $f \in \mathcal{C}_1$ has a *domain* and a *codomain*, both objects $\mathcal{C}_0$. We write $f : A \to B$ for an arrow $f \in \mathcal{C}_1$ with a domain $A \in \mathcal{C}_0$ and a codomain $B \in \mathcal{C}_0$.

Given two arrows $f : A \to B$ and $g : B \to C$, we can *compose* them, to obtain an arrow $g \circ f = gf : A \to C$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

The composition operation, when defined, is associative, i.e. $h(gf) = (hg)f$. We additionally require for each object $A \in \mathcal{C}_0$ an arrow $\mathrm{id}_A : A \to A$ that is an identity element: $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ for any $f : A \to B$.

By $Hom_{\mathcal{C}}(A, B)$ we denote a collection of arrows with a domain A and a codomain B.

Example 1. Some promiment categories: **Set**, a category of sets and functions between them; **Grp**, a category of groups and group homomorphisms; a trivial category **1** consisting of one object and one identity arrow. The last example can be generalized as follows: pick a poset $(P, \leq)$, it induces a category with objects elements of P. The set $Hom_P(a, b)$ contains exactly one arrow if $a \leq b$, and $Hom_P(a, b) = \emptyset$ otherwise.

Finite products. We say that a category $\mathcal{C}$ has *binary products* if for every pari of objects $A, B \in \mathcal{C}_0$ there is an object $A \times B$ and arrows $\pi_1 : A \times B \to A, \pi_2 : A \times B \to B$ such that for any two arrows $f : X \to A, g : X \to B$ there is a unique arrow $\langle f, g\rangle : X \to A \times B$ such taht $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$ (see the diagram below on the left).

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, dashed] \\ 1 \end{tikzcd}$$

The definition of binary products can be generalized to n-ary products for any finite n. In case $n = 0$ we speak of a *terminal object* 1, with the following property (see the diagram on the right above): for each object X there is a unique arrow $X \to 1$.

Example 2. In **Set**, a product $A \times B$ is just a cartesian product of two sets. The terminal object is then a one-element set $1 = \{*\}$.

Lawvere's diagonal argument

Generalizing from the example of sets, we call maps $1 \to X$ *global elements* of X. In **Set** such functions precisely correspond to members of X.

We can then state when some arrow $f : A \to B$ behaves like a "surjection" on global elements.

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Definition 3. An arrow $f : A \to B$ is *point-surjectve* if for every global element $b : 1 \to B$ there is a global element $a : 1 \to A$ such that $f \circ a = b$.

Equivalently, $Hom(1, f) : Hom(1, A) \to Hom(1, B)$ is surjective.

Some categories with products support "function spaces": objects B^A, which somehow internalize arrows $A \to B$ (in **Set**: a collection of arrows $Hom(A, B)$ between sets is itself a set). For such a function space we can weaken the notion of point-surjectivity, requiring that an element of the preimage of some function g is only *extensionally* equal to g. Luckily, we can state this property without mentioning categorical exponents.

Definition 4. An arrow $f : X \times A \to Y$ is *weakly point-surjective* if for every arrow $g : X \to Y$ there is a global element $a : 1 \to A$ such that for all $x : 1 \to X$, $f \circ \langle x, a\rangle = g \circ a$:

$$\forall g \exists a \forall x (f\langle x, a\rangle = gx)$$

One can think of such f as a series of functions $f\langle -, a\rangle$ such that for each $g : X \to Y$ there is a function $f\langle -, a\rangle$ which is extensionally equal to g.

Theorem 5 (Lawvere)**.** Suppose that $f : A \times A \to B$ is weakly point-surjective. Then every map $t : B \to B$ has a fixed point, i.e. an element $x : 1 \to B$ such that $tx = x$.

Proof. Consider a composite $t \circ f \circ \langle \mathrm{id}_A, \mathrm{id}_A\rangle : A \to B$.

$$\begin{tikzcd} A \times A \arrow[r, "f"] & B \arrow[d, "t"] \\ A \arrow[u, "\Delta"] \arrow[r, dotted, "t \circ f \circ \Delta"'] & B \end{tikzcd}$$

By the assumption, there is a global element $a : 1 \to A$ such that

$$\forall (x : 1 \to A).(f\langle x, a\rangle = t \circ f \circ \langle \mathrm{id}_A, \mathrm{id}_A\rangle \circ x = t(f\langle x, x\rangle)$$

In particular, for $x = a$: $f\langle a, a\rangle = t(f\langle a, a\rangle)$. Hence, $f\langle a, a\rangle$ is a fixed point of t. □

Corollary 6. Suppose that a map $\neg : \Omega \to \Omega$ doesn't have a fixed point. Then there is no weakly point-surjective map $A \to \Omega^A$ for any A.

Then we can obtain Cantor's theorem in a straightforward way: since the negation map $\neg : 2 \to 2$ has a fixed-point, there is not surjective map $A \to 2^A = \mathcal{P}(A)$. By substituting Ω for 2 we obtain Cantor's theorem in an arbitrary (non-degenerate) topos.

Russel's Paradox and Unbounded Comprehension

Suppose there is a set-theoretic universe $\mathcal{U} \in$ **Set**, a "set of all sets". To recover Russel's paradox we consider a relation $\epsilon : \mathcal{U} \times \mathcal{U} \to 2$ where $\epsilon(x, y) = 1 \iff x \in y$, and take the negation of the diagonal of ϵ:

$$\begin{tikzcd} \mathcal{U} \times \mathcal{U} \arrow[r, "\epsilon"] & 2 \arrow[d, "\neg"] \\ \mathcal{U} \arrow[u, "\Delta"] \arrow[r, dotted, "\neg \circ \epsilon \circ \Delta"'] & 2 \end{tikzcd}$$

The composite $\neg \circ \epsilon \circ \Delta$ is a map $\mathcal{U} \to 2$, that is, a predicate on $\mathcal{U}$ that is `true` on the sets x for which $\neg(x \epsilon x)$ holds; i.e. for sets that do not contain themselves. Now, for obtaining Russel's paradox we would have to show that ϵ is weakly-point surjective. What does it mean for $\mathcal{U}$ specifically? It would mean that for any predicate $\phi : \mathcal{U} \to 2$ on sets there exists a set $x \in \mathcal{U}$ (corresponding to a map $x : 1 \to \mathcal{U}$) such that the members of x are exactly such sets that satisfy ϕ:

$$\frac{\phi : \mathcal{U} \to 2}{\exists x \in \mathcal{U} \forall y \in \mathcal{U} (y \epsilon x = \phi(y))}$$

This rule is exactly the *unbounded comprehension scheme* for $\mathcal{U}$! As you can see, employing Lawvere's analysis for this paradox pinpoints exactly to the problematic part: the unbounded comprehension schema for $\mathcal{U}$. Restricting the comprehension schema to already-defined sets is exactly the fix that was utilized in axiomatic set theory. Notice that this analysis shows that the issue does not lie in self-reference or the size of $\mathcal{U}$ per se. After all, the universe $\mathcal{U}$ does not have to contain "all" sets; we can replace the word "set" in the previous paragraph by "$\mathcal{U}$-set" and the argument would still go through.

Lindenbaum-Tarski Categories and incompleteness

Consider a first-order theory $\mathbb{T}$. We form $\mathcal{C}(\mathbb{T})$ a *classifyng category* of $\mathbb{T}$ in the following way: objects of $\mathcal{C}(\mathbb{T})$ are generated by a sort object A (more object if the theory is multi-sorted), and a dummy object 2, by closure under products. Thus, the objects of $\mathbb{T}$ are of the form $A^n \times 2^m$. A map $\varphi : A^n \to 2$ is an equivalence class of provably equivalent formulas φ of n variables. A map $A^n \to 2 \times 2$ is a tuple of formulas of n free variables, and so on. A map $t : A^n \to A$ is a class of provably equal terms with n free variables. In particular, maps $1 \to 2$ are *sentences* of $\mathbb{T}$, and maps $1 \to A$ are definable constants/terms of $\mathbb{T}$.

A theory is *consistent* if the collection of maps $1 \to 2$ contains at least two elements $\mathsf{true}, \mathsf{false}$, corresponding to statements that are provable in the theory, and statements that are refutable in the theory. A theory is *complete* if the collection of maps $1 \to 2$ is exactly $\{\mathsf{true}, \mathsf{false}\}$, i.e. every sentence is either provable or refutable.

Undefinability of sat. Suppose that the satisfiabilty predicate is definable in $\mathbb{T}$:

$$\vdash \mathsf{sat}(a, \ulcorner\varphi\urcorner) \leftrightarrow \varphi(a)$$

for all φ, a.

In categorical terms, we have a Gödel encoding, $\ulcorner - \urcorner : Hom(A^n, 2) \to Hom(1, A)$, and a formula $sat : A \times A \to 2$, such that for any $\varphi : A \to 2$, and for all $a : 1 \to A$, $sat\langle a, \ulcorner\varphi\urcorner\rangle = \varphi a$. But this is exactly the condition for weak point-surjectivity! Hence, every function $2 \to 2$ has a fixed point, and we are in an inconsistent theory.

Undefinability of truth. We say that truth is definable in a theory, if there is a formula T, such that

$$\vdash T(\ulcorner\varphi\urcorner) \leftrightarrow \varphi$$

So it is very much like sat, but only for sentences. Categorically, we can say that $T : A \to 2$ is a truth predicate, if $Hom(1, T) : Hom(1, A) \to Hom(1, 2)$ is a retract of $\ulcorner - \urcorner : Hom(1, 2) \to Hom(1, A)$; or, $T \circ \ulcorner\varphi\urcorner = \varphi$. So, suppose that $\mathbb{T}$ has a truth predicate, and suppose further that it supports "substitution":

$$\mathbb{T} \vdash \mathsf{subst}(a, \ulcorner\varphi\urcorner) = \ulcorner\varphi(a)\urcorner$$

In that case, we can define sat as the composite $T \circ \mathsf{subst}$.

Incompleteness. A provability predicate is a predicate P such that

$$\mathbb{T} \vdash P(\ulcorner\varphi\urcorner) \quad \text{iff} \quad \mathbb{T} \vdash \varphi$$

In categorical terms, $P \circ \ulcorner\varphi\urcorner = \varphi$ given that both $P \circ \ulcorner\varphi\urcorner$ and φ take value in $\{\mathsf{true}, \mathsf{false}\}$. But if $\mathbb{T}$ is complete, then the provability predicate is also a truth predicate.

Assemblies and the halting problem

Consider the category *Asm* of *assemblies*. The objects are pairs $(X, \Vdash_X)$ where $X \in \mathbf{Set}$ and $\Vdash_X \subseteq \mathbb{N} \times X$ such that for each $x \in X$ there is at least one number $n \Vdash_X x$. Elements m such that $m \Vdash_X x$ are called *realizers* of x and we say that m *realizes* x. A map $f : (X, \Vdash_X) \to (Y, \Vdash_Y)$ is a morphism of assemblies if there is a partial computable function ϕ such that whenever $n \Vdash_X x$, $\phi(n)$ terminates and $\phi(n) \Vdash_Y f(y)$. We say that ϕ *tracks* or *realizes* f. The products in *Asm* are given by surjective pairings. There is a natural numbers object $\mathbf{N}$ in *Asm* given by $(\mathbb{N}, \Vdash_N)$ where $n \Vdash_N m$ iff $n = m$.

Proposition 7. The morphisms $\mathbf{N} \to \mathbf{N}$ are exactly (total) computable functions.

Definition 8. *Asm* has all finite types. For instance, the object 2 is given by $(\{0, 1\}, \Vdash_2)$ where $i \Vdash_2 j$ iff $i = j$.

Suppose that the halting problem is decidable. We define a morphism $halt : \mathbf{N} \times \mathbf{N} \to 2$ such that $halt(n, m) = 1$ iff the partial computable function $\{n\}(-) : \mathbb{N} \rightharpoonup \mathbb{N}$ terminates on the input m. For *halt* to be weak point-surjective we must show that for any morphism $f : \mathbf{N} \to 2$ there is a number n such that $halt(n, m) = f(m)$ for all m, i.e. $\{n\}(m)$ terminates iff $f(m) = 1$. How do we construct such n? Well, f is tracked by some computable ϕ, so n is just the Gödel code of an algorithm/function that runs $\phi(m)$ on input m and terminates if the output of $\phi(m)$ is 1, and diverges otherwise.

Obtaining fixed points

Retractions & the Y-combinator. An epimorphism $r : E \to B$ is said to be *split*, if there is a map $s : B \to E$ in the opposite direction such that $r \circ s = \mathrm{id}_B$. This is equivalent to saying that $Hom(A, r) : Hom(A, E) \to Hom(A, B)$ is surjective for all A. Clearly, any split epimorphism is point-surjective, the choice for the witness for the existential quantifier is given by s. (However, not every epimorphism is point-surjective, and not every point-surjective map is epi)

Consider the category $CPO_\perp$ of direct-complete partial orders with $\perp$. It is a cartesian closed category with a *reflexive* element U; that is an object $U \neq 1$ such that there is a retraction $r : U \to U^U$. Such a domain U provides a model for untyped λ-calculus; furthermore, a complete class of models of λ-calculus arises in such a way: see section 5.5 in Barendregt's book.

Anyway, what follows is that every map $t : U \to U$ has a fixed point; this fixed point is exactly the one given by the Y-combinator!

By computation, a fixed point of t is given by $\overline{r} \circ \Delta \circ s(\overline{t \circ \overline{r} \circ \Delta})$. Mixing syntax and semantics informally we have $\overline{r} \circ \langle a, b\rangle = ab$ and $s(x \mapsto g(x)) = \lambda x.g(x)$, so the fixed point is

$$(s(\overline{t \circ \overline{r} \circ \Delta}))(s(\overline{t \circ \overline{r} \circ \Delta})) = (\lambda x.(t \circ \overline{r} \circ \langle x, x\rangle))(\lambda x.(t \circ \overline{r} \circ \langle x, x\rangle)) = (\lambda x.(t(xx)))(\lambda x.(t(xx)))$$

which is exactly $Y(t)$.

Enumerations of r.e. sets. Consider an assembly $\Sigma \in Asm$ defined as an underlying set $\{\top, \perp\}$ with the realizability relation

$$n \Vdash \top \iff \{n\}(n)\downarrow \qquad n \Vdash \perp \iff \{n\}(n)\uparrow$$

Such Σ is called a *r.e. subobject classifier* or a *r.e. dominance*.

Morphisms $X : \mathbf{N} \to \Sigma$ are recursively-enumerable sets. Given a map $X : \mathbf{N} \to \Sigma$ tracked by ϕ we define a set $\overline{X} = \{x \in \mathbb{N} \mid X(x) = \top\}$. To check that $n \in \overline{X}$ we attempt to compute $\{\phi(n)\}(\phi(n))$. If $\{\phi(n)\}(\phi(n))$ terminates, then $n \in \overline{X}$. Similarly, given a r.e. set Y we put $\overline{Y}(n) = \top \iff (n \in Y)$; $\overline{Y}$ is then tracked by a computable function that sends n to the Gödel code of the decision procedure $x \mapsto [n \in Y]$.

The exponent $\Sigma^{\mathbf{N}}$ is then the collection of r.e. sets. We know that there is an enumeration of r.e. sets, thus a weakly point-surjective $W : \mathbf{N} \to \Sigma^{\mathbf{N}}$. Hence, by Lawvere's theorem every map $\Sigma \to \Sigma$ has a fixed point. It immediately follows that negation is not definable on Σ and hence r.e. sets are not closed under complements.

Note that $\Sigma^{\mathbf{N}} \simeq \Sigma^{\mathbf{N}\times\mathbf{N}} \simeq \Sigma^{\mathbf{N}^{\mathbf{N}}}$, so every map $\Sigma^{\mathbf{N}} \to \Sigma^{N}$ has a fixed point as well. We can identify the exponent Σ^{N} with an assembly (RE, W) where RE is the set of r.e. subsets of $\mathbf{N}$ and $W(A) = \{e \mid A = W_e\}$ for an enumeration $\{W_i\}_i$ of r.e. sets.

A map $F : (RE, W) \to (RE, W)$ is an enumeration operator: $F(W_e) = W_{\phi(e)}$, for some computable ϕ. The Lawvere's argument states that every such operator has a fixed point: $W_k = W_{\phi(k)}$. Consider a computable ϕ which for every n outputs the r.e. index (todo: is that the term?) of a r.e. set that is just a singleton $\{n\}$, that is $W_{\phi(n)} = \{n\}$. By the existence of a fixed point we have a number k such that $W_k = \{k\}$.

References

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Appendix

We would like to make a few additional remarks.

Remark 9. A finer analysis of the argument might reveal the following fact: it is not necessary to take the diagonal map $\Delta : A \to A \times A$. One can easily take any other map $\langle \mathrm{id}_A, k\rangle$ for a "good" $k : A \to A$ (say, if k is an isomorphism). Then the fixed-point for a map $t : B \to B$ can be constructed from

$$t(f\langle x, k(x)\rangle) = f\langle x, b\rangle$$

If k is an isomorphism, then we can find such x that $k(x) = b$. Then we obtain the fixed point in a similar manner.