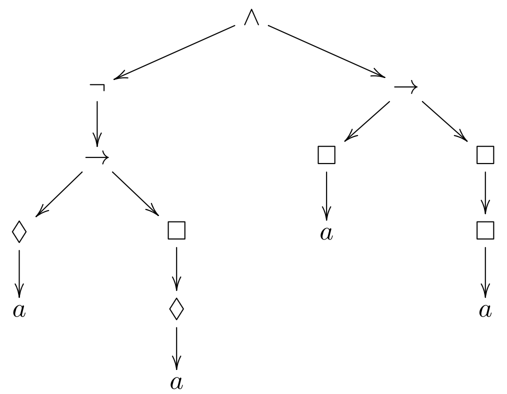
Formal Reasoning 2016
Solutions Test Block 5: Modal Logic
(19/12/16)

1. Draw a tree according to the structure of the modal formula:

$$\neg(\Diamond a \to \Box\Diamond a) \land (\Box a \to \Box\Box a)$$



2. In this exercise we use the dictionary:

R	It rains
W	I get wet

Give an English sentence that approximates the meaning of the formula from doxastic logic

$$\Box(R \to W) \to \Box(\Box R \to \Box W)$$

as well as possible.

If I believe that if it rains I get wet, then I believe that if I believe it rains that I believe that I get wet.

3. Give a Kripke model in which the formula

$$\Box(R \to W) \to \Box(\Box R \to \Box W)$$

is false, and explain your answer.

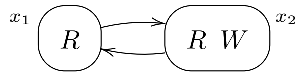
So we have to find a model $\mathcal{M}_3$ that contains a world x such that:

- $x \Vdash \Box(R \to W)$
- $x \nVdash \Box(\Box R \to \Box W)$, which means that there exists a world y such that
  - $y \in R(x)$ and
  - $y \Vdash \Box R$ and
  - $y \nVdash \Box W$, which means that there exists a wold z such that
    - $z \in R(y)$ and

- $z \not\Vdash W$

Take for instance:

$\mathcal{M}_3$:



Or in the more formal notation $\mathcal{M}_3 = \langle W, R, V \rangle$, where $W = \{x_1, x_2\}$, and R is defined by $R(x_1) = \{x_2\}$ and $R(x_2) = \{x_1\}$, and V is defined by $V(x_1) = \{R\}$ and $V(x_2) = \{R, W\}$.

Now if for x and z we take x_1 and for y we take x_2, we see that this is an instance of the requirements given above.

We can also use a table to prove that our model is correct. Let us abbreviate our formula: $f_3 := \Box(R \to W) \to \Box(\Box R \to \Box W)$. Then:

$\Vdash$	R	W	$R \to W$	$\Box(R \to W)$	$\Box R$	$\Box W$	$\Box R \to \Box W$	$\Box(\Box R \to \Box W)$	f_3
x_1	1	0	0	1	1	1	1	0	0
x_2	1	1	1	0	1	0	0	1	1

In the table we can see that $x_1 \not\Vdash f_3$. But this means automatically that $\mathcal{M}_3 \not\models f_3$.

4. Give an LTL formula that states that first after some non-zero amount of time a will be true, then at some point in time after that b will be true, and finally some time after that c will be true.

   Take for instance the formula f_4:

$$\mathcal{XF}(a \land \mathcal{XF}(b \land \mathcal{XF}c))$$

5. (a) Axiom scheme 4 is:

$$\Box f \to \Box\Box f$$

   Are all instances of axiom scheme 4 true in all LTL models?

   If this is the case, explain why. If this is not the case, then give an instance of the scheme as well as an LTL model that together form a counterexample, and explain why.

   Yes, this is the case. If we assume that $\Box f$ holds in an LTL Kripke model, we know that $x_i \Vdash f$ for all $i \in \mathbb{N}$, because $\Box$ is $\mathcal{G}$. We now have to prove $\Box\Box f$. This means that we have to prove that for all $k \in \mathbb{N}$, $x_k \Vdash \Box f$. But this means that we have to prove that for all $k \in \mathbb{N}$ and for all $l \in \mathbb{N}$ it holds that if $l \geq k$, then $x_l \Vdash f$. But we already know that $x_l \Vdash f$ for all $l \in \mathbb{N}$, so it must certainly hold for those $l \in \mathbb{N}$ that comply to the extra requirement that $l \geq k$. So the axiomscheme $\Box f \to \Box\Box f$ holds in any LTL Kripke model.

   (b) Axiom scheme 5 is:

$$\Diamond f \to \Box\Diamond f$$

   Are all instances of axiom scheme 5 true in all LTL models?

   If this is the case, explain why. If this is not the case, then give an instance of the scheme as well as an LTL model that together form a counterexample, and explain why.

No, this is not the case. Consider the model $\mathcal{M}_5 = \langle W, R, V \rangle$, where $W = \{x_i \,|\, i \in \mathbb{N}\}$ and R is given by $R(x_i) = \{x_j \,|\, j \in \mathbb{N} \text{ and } j \geq i\}$ and V is defined by $V(x_0) = \{a\}$ and $V(x_i) = \emptyset$ for all $i \in \mathbb{N}$ and $i \geq 1$. From the definition of W and R it follows that this is indeed an LTL Kripke model.

Because $a \in V(x_0)$ and $x_0 \in R(x_0)$ we have that $x_0 \Vdash \Diamond a$. However $x_0 \Vdash \Box\Diamond a$ does not hold. Obviously $x_1 \in R(x_0)$. So $x_1 \Vdash \Diamond a$ should hold. However, since $R(x_1) = \{x_i \,|\, i \in \mathbb{N} \text{ and } i \geq 1\}$ and $V(x_i) = \emptyset$ for all $i \in \mathbb{N}$ such that $i \geq 1$, it is clear that $\Diamond a$ does not hold in x_1 and therefore $\Box\Diamond a$ does not hold in x_0. And hence $\Diamond a \to \Box\Diamond a$ does not hold in x_0 and therefore $\Diamond a \to \Box\Diamond a$ does not hold in model $\mathcal{M}_5$. So this is indeed an instance of the axiomscheme that does not hold in all LTL Kripke models.