program extraction

type theory

week 5

2006 03 13

why intuitionism?

foundational crisis

Russell, start 20th century:

$$\{x \mid x \notin x\} \in \{x \mid x \notin x\}\ ?$$

shows that naive set theory / type theory is inconsistent

Brouwer

three schools:

- **formalism**

  Hilbert . . . leads eventually to ZFC set theory

- **logicism**

  Russell . . . early version of type theory

- **intuitionism**

  Brouwer rejects excluded middle, proves all functions continuous

  ↓

  Heyting the logic of intuitionism

  ↓

  Bishop variant that is strictly weaker than classical mathematics

constructivism

Brouwer-Heyting-Kolmogorov interpretation

proof of $\bot$	$\ldots$	doesn't exist
proof of $A \to B$	$\leftrightarrow$	function that maps proofs of A to proofs of B
proof of $A \land B$	$\leftrightarrow$	pair of a proof of A and a proof of B
proof of $A \lor B$	$\leftrightarrow$	either a proof of A or a proof of B
proof of $\forall x.\, P(x)$	$\leftrightarrow$	function that maps object x to proof of $P(x)$
proof of $\exists x.\, P(x)$	$\leftrightarrow$	object a together with proof of $P(a)$

proof of existence corresponds to constructing an example

proofs are programs

program extraction

intuitionistic proof

$\updownarrow$

executable algorithm

intuitionism the natural logic for computer science?

‘code-carrying proofs’

verified programs

two approaches

- **correctness proofs**

  program $\longrightarrow$ $\ldots + $ proof

- **program extraction**

  program $\longleftarrow$ proof

Hoare logic

imperative program

↓

annotated imperative program

formulas of predicate logic

↓

proof obligations

why & caduceus

Jean-Christophe Filliâtre

- **why**

  Hoare logic for small programming language

  union of imperative and functional programming language

programming language independent

proof assistant independent

designed to be used with Coq

- **caduceus**

  Hoare logic for almost full ANSI C

  built on top of why

example

```
/*@ requires \valid_range(t,0,n-1)
  @ ensures
  @   (0 <= \result < n => t[\result] == v) &&
  @   (\result == n => \forall int i; 0 <= i < n => t[i] != v)
  @*/
int index(int t[], int n, int v) {
  int i = 0;
  /*@ invariant 0 <= i && \forall int k; 0 <= k < i => t[k] != v
    @ variant n - i */
  while (i < n) {
    if (t[i] == v) break;
    i++;
  }
  return i;
}
```

program extraction

specification

↓

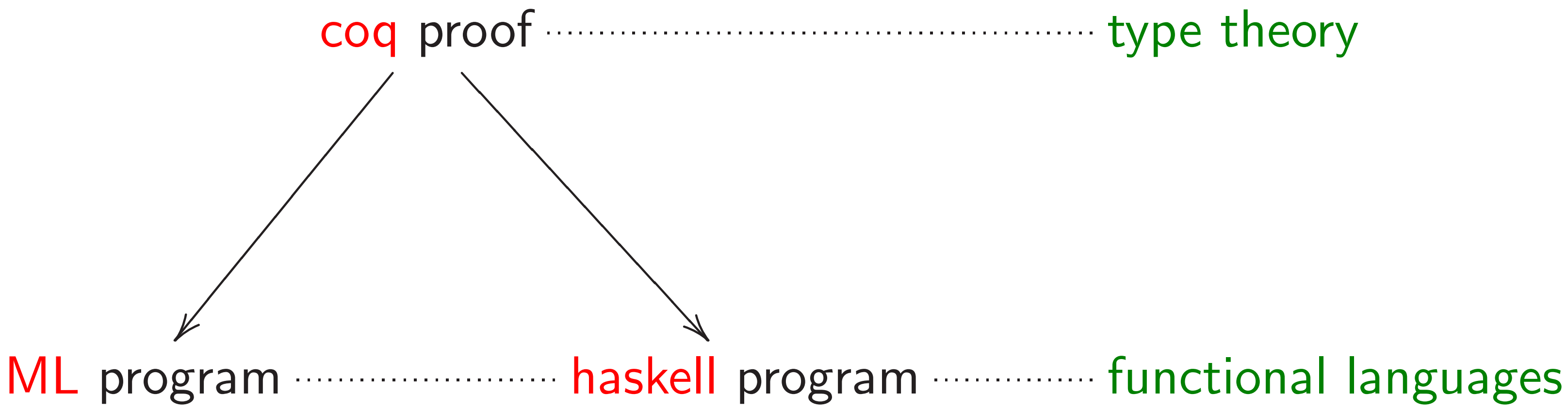
constructive proof of existence of solution to the specification

↓

automatically generated functional program

guaranteed correct with respect to the specification

extraction to functional programs



coq proof ······ type theory

ML program ······ haskell program ······ functional languages

example: mirroring trees

`bintree`

inductive type

```
Inductive bintree : Set :=
  leaf : nat -> bintree
| node : bintree -> bintree -> bintree.
```

mirror

recursive function

```
Fixpoint mirror (t : bintree) : bintree :=
  match t with
    leaf n => leaf n
  | node t1 t2 => node (mirror t2) (mirror t1)
  end.
```

Mirrored

inductive predicate

```
Inductive Mirrored : bintree -> bintree -> Prop :=
  Mirrored_leaf :
    forall n : nat, Mirrored (leaf n) (leaf n)
| Mirrored_node :
    forall t1 t2 t1' t2' : bintree,
      Mirrored t1 t1' -> Mirrored t2 t2' ->
      Mirrored (node t1 t2) (node t2' t1').
```

correctness of `mirror`

```
Lemma Mirrored_mirror :
  forall t : bintree, Mirrored t (mirror t).
induction t.
simpl.
apply Mirrored_leaf.
simpl.
apply Mirrored_node.
exact IHt1.
exact IHt2.
Qed.
```

two kinds of existential statements

$$\exists x : A.\, P(x)$$

- existential in `Prop`

```
exists x : A, P x
```

- existential in `Set`

```
{x : A | P x}
```

definition of ex

inductive type

```
Inductive  ex (A : Set) (P : A -> Prop) : Prop :=
  ex_intro : forall x : A, P x -> ex P.
```

in practice

```
exists x : A. P x
```

is syntax for

```
ex A (fun x : A => P x)
```

definition of `sig`

inductive type

```
Inductive sig (A : Set) (P : A -> Prop) : Set :=
  exist : forall x : A, P x -> sig P.
```

in practice

```
{x : A | P x}
```

is syntax for

```
sig A (fun x : A => P x)
```

existence proof for specification

```
Lemma Mirror :
  forall t : bintree, {t' : bintree | Mirrored t t'}.
induction t.
exists (leaf n).
apply Mirrored_leaf.
elim IHt1.
intros t1' H1.
elim IHt2.
intros t2' H2.
exists (node t2' t1').
apply Mirrored_node.
exact H1.
exact H2.
Qed.
```

extracting the program

```
Coq < Extraction Mirror.
(** val mirror : bintree -> bintree sig0 **)

let rec mirror = function
  | Leaf n -> Leaf n
  | Node (b0, b1) -> Node ((mirror b1), (mirror b0))

Coq <



type 'a sig0 = 'a
```

summarizing

- **specification**

  ```
  Inductive Mirrored : bintree -> bintree -> Prop := ...
  ```

- **implementation**

  ```
  Fixpoint mirror (t : bintree) : bintree := ...
  ```

- **correctness**

  ```
  forall t : bintree, Mirrored t (mirror t)
  ```

- **program extracted from existence proof for specification**

  ```
  forall t : bintree, {t' : bintree | Mirrored t t'}
  ```

the general pattern

Π_2 sentences

program specification

$$\forall x : A.\, P(x) \to \exists y : B.\, Q(x, y)$$

A	input type
B	output type
$P(x)$	precondition
$Q(x, y)$	input/output behavior

the proof term versus the extracted program

coq type theory = functional programming language
coq proof term = functional program

ML language = functional programming language
ML program = functional program

program extraction is **almost** the identity function

- differences in type system
- not all parts of coq terms are computationally relevant

`Prop` versus `Set`

not all coq terms are computationally relevant

‘Curry-Howard-de Bruijn’ terms don’t need to be calculated

terms of type in `Prop`	‘non-informative’	discarded
terms of type in `Set`	‘informative’	kept

‘elimination of Prop over Set’

```
Inductive or (A : Prop) (B : Prop) : Prop :=
  or_introl : A -> A \/ B
| or_intror : B -> A \/ B.

Definition foo (A : Prop) (H : A \/ ~A) : bool :=
  match H with
    or_introl _ => true
  | or_intror _ => false
  end.
```

Elimination of an inductive object of sort : ‘`Prop`’
is not allowed on a predicate in sort : ‘`Set`’
because non-informative objects may not construct informative ones.

example: negation in the booleans

statement

```
forall b : bool, {b' : bool | ~(b = b')}
```

extracted program

```
(** val negation : bool -> bool sig0 **)

let negation = function
  | True -> False
  | False -> True
```

proof term

```
fun b : bool =>
bool_rec (fun b0 : bool => b' : bool | b0 <> b')
  (exist (fun b' : bool => true <> b') false
     (fun H : true = false =>
      let H0 :=
        eq_ind true (fun ee : bool => if ee return Prop then True else False)
          I false H in
      False_ind False H0))
  (exist (fun b' : bool => false <> b') true
     (fun H : false = true =>
      let H0 :=
        eq_ind false
          (fun ee : bool => if ee return Prop then False else True) I true H in
      False_ind False H0)) b
```

```
bool_rec :
forall P : bool -> Set, P true -> P false -> forall b : bool, P b
```

example: the predecessor function

statement

```
forall n : nat, ~(n = 0) -> {m : nat | S m = n}
```

extracted program

```
(** val pred : nat -> nat sig0 **)

let rec pred = function
  | O -> assert false (* absurd case *)
  | S n0 -> n0
```

the `assert` corresponds in the proof term to . . .

```
False_rec {m : nat | S m = O} (H (refl_equal O))
      : {m : nat | S m = O}
```

. . . recursion on a proof of `False`

FTA project

coq formalization of non-trivial mathematical theorem

Fundamental Theorem of Algebra
every non-constant complex polynomial has a root

finished in 2000
Herman Geuvers, Randy Pollack, Freek Wiedijk, Jan Zwanenburg

intuitionistic proof

extracting the Fundamental Theorem of Algebra

complex polynomials

$$\forall p.\, (p \text{ not constant}) \to \exists z.\, p(z) = 0$$

↓ program extraction

program for calculating roots of polynomials

input	complex polynomial
output	sequence converging to a root

extracting the Intermediate Value Theorem

real polynomials

$$\forall p.\,(p(0) < 0 \wedge p(1) > 0) \to \exists x.\,(0 < x \wedge x < 1 \wedge p(x) = 0)$$

take $p(x) = x^2 - 2$

↓ program extraction

program for approximating $\sqrt{2}$

example: sorting lists

`natlist`

inductive type

```
Inductive natlist : Set :=
  nil : natlist
| cons : nat -> natlist -> natlist.
```

Sorted

inductive predicate

```
Inductive Sorted : natlist -> Prop :=
  Sorted_nil : Sorted nil
| Sorted_one : forall n : nat, Sorted (cons n nil)
| Sorted_cons :
    forall (n m : nat) (l : natlist),
    n <= m -> Sorted (cons m l) -> Sorted (cons n (cons m l)).
```

Inserted

inductive predicate

```
Inductive Inserted (n : nat) : natlist -> natlist -> Prop :=
  Inserted_front :
    forall l : natlist, Inserted n l (cons n l)
| Inserted_cons :
    forall (m : nat) (l l' : natlist),
    Inserted n l l' -> Inserted n (cons m l) (cons m l').
```

`Inserted` 4 [1,2,3] [4,1,2,3]

`Inserted` 4 [1,2,3] [1,4,2,3]

`Inserted` 4 [1,2,3] [1,2,4,3]

`Inserted` 4 [1,2,3] [1,2,3,4]

Permutation

inductive predicate

```
Inductive Permutation : natlist -> natlist -> Prop :=
  Permutation_nil : Permutation nil nil
| Permutation_cons :
    forall (n : nat) (l l' l'' : natlist),
    Permutation l l' -> Inserted n l' l'' ->
    Permutation (cons n l) l''.
```

statement

```
forall l : natlist,
{l' : natlist | Permutation l l' /\ Sorted l'}
```

insert

recursive function

```
Fixpoint insert (n : nat) (l : natlist) {struct l} : natlist :=
  match l with
    nil => cons n nil
  | cons m k =>
      match le_lt_dec n m with
        left _ => cons n (cons m k)
      | right _ => cons m (insert n k)
      end
  end.
```

sort

recursive function

```
Fixpoint sort (l : natlist) : natlist :=
  match l with
    nil => nil
  | cons m k => insert m (sort k)
  end.
```