Secondly, order is antisymmetric: 5 is bigger than 3 but 3 is not bigger than 5. It is on these two properties—transitivity and antisymmetry—that the theory of order rests.

Order relations are of two types: strict and non-strict. Outside mathematics, the strict notion is more common. The statement 'Charles is taller than Bruce' is generally taken to mean 'Charles is strictly taller than Bruce', with the possibility that Charles is the same height as Bruce not included. Mathematicians usually allow equality and write, for instance, $3 \leqslant 3$ and $3 \leqslant 22/7$. We shall deal mainly with non-strict order relations.

Finally a comment about comparability. Statement (f) asserts that, for the ordering $<$ on the real numbers, any two distinct elements can be compared. This property is possessed by many familiar orderings, but it is not universal. For example, there certainly exist human beings A and B such that A is not a descendant of B and B is not a descendant of A. Non-comparability also arises in (e).

1.2 Definitions. Let P be a set. An **order** (or **partial order**) on P is a binary relation $\leqslant$ on P such that, for all $x, y, z \in P$,

(i) $x \leqslant x$,

(ii) $x \leqslant y$ and $y \leqslant x$ imply $x = y$,

(iii) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$.

These conditions are referred to, respectively, as **reflexivity**, **antisymmetry** and **transitivity**. A set P equipped with an order relation $\leqslant$ is said to be an **ordered set** (or **partially ordered set**). Some authors use the shorthand **poset**. Usually we shall be a little slovenly and say simply 'P is an ordered set'. Where it is necessary to specify the order relation overtly we write $\langle P; \leqslant \rangle$.

An order relation $\leqslant$ on P gives rise to a relation $<$ of **strict inequality**: $x < y$ in P if and only if $x \leqslant y$ and $x \neq y$. It is possible to re-state conditions (i)–(iii) above in terms of $<$, and so to regard $<$ rather than $\leqslant$ as the fundamental relation; see Exercise 1.1.

Other notation associated with $\leqslant$ is predictable. We use $x \leqslant y$ and $y \geqslant x$ interchangeably, and write $x \nleqslant y$ to mean '$x \leqslant y$ is false', and so on. Less familiar is the symbol $\parallel$ used to denote non-comparability: we write $x \parallel y$ if $x \nleqslant y$ and $y \nleqslant x$.

We later deal systematically with the construction of new ordered sets from existing ones. However there is one such construction which it is convenient to have available immediately. Let P be an ordered set and let Q be a subset of P. Then Q inherits an order relation from P; given $x, y \in Q$, $x \leqslant y$ in Q if and only if $x \leqslant y$ in P. We say in these circumstances that Q has the order **induced from** P.

1.3 Chains and antichains. Let P be an ordered set. Then P is a **chain** if, for all $x, y \in P$, either $x \leqslant y$ or $y \leqslant x$ (that is, if any two elements of P are comparable). Alternative names for a chain are **linearly ordered set** and **totally ordered set**. At the opposite extreme from a chain is an antichain. The ordered set P is an **antichain** if $x \leqslant y$ in P only if $x = y$. Clearly, with the induced order, any subset of a chain (an antichain) is a chain (an antichain).

Let P be the n-element set $\{0, 1, \dots, n-1\}$. We write $\mathbf{n}$ to denote the chain obtained by giving P the order in which $0 < 1 < \cdots < n-1$ and $\overline{\mathbf{n}}$ for P regarded as an antichain. Any set S may be converted into an ordered set $\overline{S}$ by giving S the antichain order.

Examples from mathematics, computer science and social science

We hinted in 1.1 at a variety of situations in which order structure is present. In 1.2 we developed the vocabulary for treating these examples systematically. This section is a catalogue of ordered sets, drawn from mathematics, computer science and the social sciences.

1.4 Number systems. The set of real numbers, $\mathbb{R}$, forms a chain in its usual order. Each of $\mathbb{N}$ (the natural numbers $\{1, 2, 3, \dots\}$), $\mathbb{Z}$ (the integers) and $\mathbb{Q}$ (the rational numbers) also has a natural order making it a chain. In each case this order relation is compatible with the arithmetic structure in the sense that the sum and product of two elements strictly greater than zero is also greater than zero. On the other hand, the complex numbers, $\mathbb{C}$, carry no order relation with this compatibility property; see Exercise 1.24.

We denote the set $\mathbb{N} \cup \{0\}$ $(= \{0, 1, 2, \dots\})$ by $\mathbb{N}_0$. Endowed with the order in which $0 < 1 < 2 < \dots$, the set $\mathbb{N}_0$ becomes the chain known in set theory as ω. A different order on $\mathbb{N}_0$ is defined as follows. Write $m \preccurlyeq n$ if and only if there exists $k \in \mathbb{N}_0$ such that $km = n$ (that is, m divides n). Then $\preccurlyeq$ is an order relation. Of course, $\langle \mathbb{N}_0; \preccurlyeq \rangle$ is not a chain. Yet another order on $\mathbb{N}_0$ is introduced in 1.24 for use in Chapters 3 and 4.

1.5 Families of sets. Let X be any set. The powerset $\wp(X)$, consisting of all subsets of X, is ordered by set inclusion: for $A, B \in \wp(X)$, we define $A \leqslant B$ if and only if $A \subseteq B$.

Any subset of $\wp(X)$ inherits the inclusion order. Such a family of sets might be specified set-theoretically. For example, it might consist

Figure 1.2(v) gives a diagram for the subset of Σ^* consisting of strings of length not more than 3.

Maps between ordered sets

This section introduces structure-preserving maps between ordered sets. In particular, it provides the machinery for deciding when two ordered sets are essentially the same.

1.11 Definitions. Let P and Q be ordered sets. A map $\varphi\colon P \to Q$ is said to be

(i) **order-preserving** (or, alternatively, **monotone**) if $x \leqslant y$ in P implies $\varphi(x) \leqslant \varphi(y)$ in Q;

(ii) an **order-embedding** if $x \leqslant y$ in P if and only if $\varphi(x) \leqslant \varphi(y)$ in Q;

(iii) an **order-isomorphism** if it is an order-embedding mapping P onto Q.

When $\varphi\colon P \to Q$ is an order-embedding we write $\varphi\colon P \hookrightarrow Q$. When there exists an order-isomorphism from P to Q, we say that P and Q are **order-isomorphic** and write $P \cong Q$.

1.12 Examples. Figure 1.3 shows some maps between ordered sets. The map φ_1 is not order-preserving. Each of φ_2–φ_5 is order-preserving, but not an order-embedding. The map φ_6 is an order-embedding, but not an order-isomorphism.

1.13 Remarks. The following are all easy to prove.

(1) Let $\varphi\colon P \to Q$ and $\psi\colon Q \to R$ be order-preserving maps. Then the composite map $\psi \circ \varphi$, given by $(\psi \circ \varphi)(x) = \psi(\varphi(x))$ for $x \in P$, is order-preserving. More generally the composite of a finite number of order-preserving maps is order-preserving, if it is defined.

(2) Let $\varphi\colon P \hookrightarrow Q$ and let $\varphi(P)$ (defined to be $\{\,\varphi(x) \mid x \in P\,\}$) be the image of φ. Then $\varphi(P) \cong P$. This justifies the use of the term embedding.

(3) An order-embedding is automatically a one-to-one map (because $\leqslant$ is reflexive on Q and antisymmetric on P). An order-isomorphism is bijective (that is, one-to-one and onto).

(4) Ordered sets P and Q are order-isomorphic if and only if there exist order-preserving maps $\varphi\colon P \to Q$ and $\psi\colon Q \to P$ such that

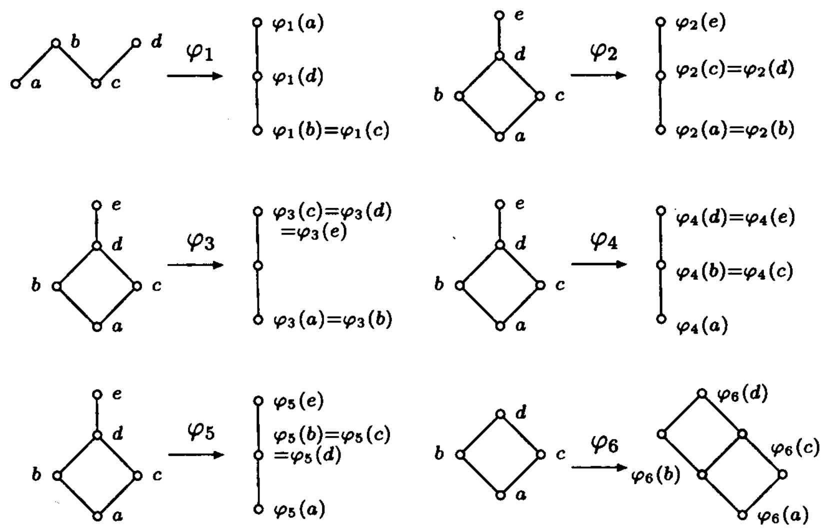


Figure 1.3

$\varphi \circ \psi = \mathrm{id}_Q$ and $\psi \circ \varphi = \mathrm{id}_P$ (where $\mathrm{id}_S\colon S \to S$ denotes the **identity map** on S given by $\mathrm{id}_S(x) = x$ for all $x \in S$.)

The diagrammatic approach to finite ordered sets is made fully legitimate by Proposition 1.15, which follows easily from Lemma 1.14.

1.14 Lemma. *Let P and Q be finite ordered sets and let $\varphi\colon P \to Q$ be a bijective map. Then the following are equivalent:*

(i) *φ is an order-isomorphism;*

(ii) *$x < y$ in P if and only if $\varphi(x) < \varphi(y)$ in Q;*

(iii) *$x \prec y$ in P if and only if $\varphi(x) \prec \varphi(y)$ in Q.*

Proof. The equivalence of (i) and (ii) is immediate from the definitions.

Now assume (ii) holds and take $x \prec y$ in P. Then $x < y$, so $\varphi(x) < \varphi(y)$ in Q. Suppose there exists $w \in Q$ such that $\varphi(x) < w < \varphi(y)$. Since φ is onto, there exists $u \in P$ such that $w = \varphi(u)$. By (ii), $x < u < y$ ↯. Hence $\varphi(x) \prec \varphi(y)$. The reverse implication is proved in much the same way. Hence (iii) holds.

Now assume (iii) and let $x < y$ in P. Then there exist elements $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$. By (iii), $\varphi(x_0) = \varphi(x) \prec \varphi(x_1) \prec \ldots \prec \varphi(x_n) = \varphi(y)$. Hence $\varphi(x) < \varphi(y)$. The reverse implication is proved similarly, using the fact that φ is onto. Hence (ii) holds. ∎

1.15 Proposition. *Two finite ordered sets P and Q are order-isomorphic if and only if they can be drawn with identical diagrams.*

Proof. Assume there exists an order-isomorphism $\varphi\colon P \to Q$. To show that the same diagram represents both P and Q, note that the diagram is determined by the covering relation and invoke 1.14 (i) $\Rightarrow$ (iii). Conversely, assume P and Q can both be represented by the same diagram, D. Then there exist bijective maps f and g from P and Q onto the points of D. The composite map $\varphi = g^{-1} \circ f$ is bijective and satisfies condition (iii) in Lemma 1.14, so is an order-isomorphism. ∎

1.16 Speaking categorically. In modern pure mathematics it is rare for a class of structures of a given type to be introduced without an associated class of structure-preserving maps following hard on its heels. Ordered sets + order-preserving maps is one example. Others are groups + group homomorphisms, vector spaces over a field + linear maps, topological spaces + continuous maps, and we later meet lattices + lattice homomorphisms, CPOs + continuous maps, etc., etc. The recognition that an appropriate unit for study is a class of objects together with its structure-preserving maps (or **morphisms**) leads to category theory. Informally, a **category** is a class of objects + morphisms, with an operation of composition of morphisms satisfying a set of natural conditions suggested by examples such as those above.

Commuting diagrams of objects and morphisms expressing properties of categories and *their* structure-preserving maps (called **functors**) form the basis of category theory. The representation theory we present in Chapters 8 and 10 is a prime example of a topic which owes its development to the apparatus of category theory. We do not have sufficient need to call on the theory of categories to warrant setting up its formalism here, but it would be wrong not to acknowledge its subliminal influence.

The Duality Principle; down-sets and up-sets

1.17 The dual of an ordered set. Given any ordered set P we can form a new ordered set P^∂ (the **dual** of P) by defining $x \leqslant y$ to hold in P^∂ if and only if $y \leqslant x$ holds in P. For P finite, we obtain a diagram for P^∂ simply by 'turning upside down' a diagram for P. Figure 1.4 provides a simple illustration.

To each statement about P there corresponds a statement about P^∂. For example, we can assert that in P in Figure 1.4 there exists a unique element covered by exactly one other element, while in P^∂ there exists a unique element covering exactly one other element. In general, given any statement Φ about ordered sets, we obtain the **dual statement** Φ^∂ by replacing each occurrence of $\leqslant$ by $\geqslant$ and vice versa.

Thus ordered set concepts and results hunt in pairs. This fact can often be used to give two theorems for the price of one or to reduce work (as, for example, in the proof of Theorem 5.2). The formal basis for this observation is the Duality Principle below; its proof is a triviality.

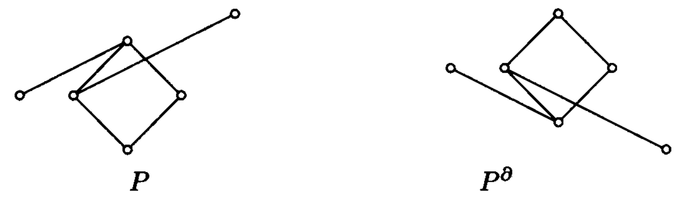


Figure 1.4

1.18 The Duality Principle. *Given a statement Φ about ordered sets which is true in all ordered sets, then the dual statement Φ^∂ is true in all ordered sets.*

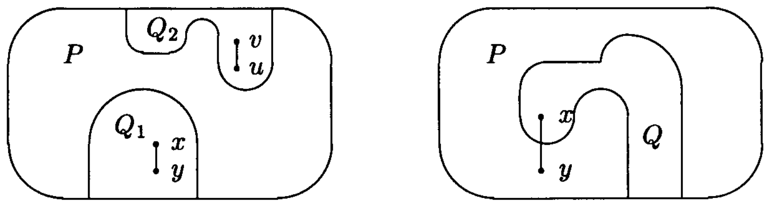
Associated with any ordered set are two important families of sets.

1.19 Definitions and remarks. Let P be an ordered set and $Q \subseteq P$.

(i) Q is a **down-set** (alternative terms include **decreasing set** or **order ideal**) if, whenever $x \in Q$, $y \in P$ and $y \leqslant x$, we have $y \in Q$.

(ii) Dually, Q is an **up-set** (alternative terms are **increasing set** or **order filter**) if, whenever $x \in Q$, $y \in P$ and $y \geqslant x$, we have $y \in Q$.

It may help to think of a down-set as one which is 'closed under going down'. Down-sets and up-sets may be depicted in a stylized way in a 'directional Venn diagram'; see Figure 1.5. Such drawings do not have the formal status of diagrams, as defined in 1.9.



Q_1 a down-set, Q_2 an up-set $\qquad$ Q not a down-set

Figure 1.5

Besides being related by duality, down-sets and up-sets are related by complementation: Q is a down-set if and only if $P \setminus Q$ is an up-set. The proof is left as an exercise.

Given an arbitrary subset Q of P and $x \in P$, we define

$$\downarrow Q = \{y \in P \mid (\exists x \in Q)\, y \leqslant x\} \quad \text{and} \quad \uparrow Q = \{y \in P \mid (\exists x \in Q)\, y \geqslant x\},$$
$$\downarrow x = \{y \in P \mid y \leqslant x\} \quad \text{and} \quad \uparrow x = \{y \in P \mid y \geqslant x\}.$$

These are read 'down Q', etc. It is easily checked that $\downarrow Q$ is the smallest down-set containing Q and that Q is a down-set if and only if $Q = \downarrow Q$, and dually for $\uparrow Q$. Clearly $\downarrow\{x\} = \downarrow x$, and dually.

The family of all down-sets of P is denoted by $\mathcal{O}(P)$. It is itself an ordered set, under the inclusion order, and plays a crucial role in later chapters. The letter $\mathcal{O}$ is traditional here; it comes from the term order ideal.

When P is finite, every non-empty set in $\mathcal{O}(P)$ is expressible in the form $\bigcup_{i=1}^{k} \downarrow x_i$ (where $\{x_1, \ldots, x_k\}$ is an antichain). This provides a recipe for finding $\mathcal{O}(P)$ (though one which is practical only when P is small).

1.20 Examples.

(1) Consider the ordered set in Figure 1.1(iii). The sets $\{c\}$, $\{a,b,c,d,e\}$ and $\{a,b,d,f\}$ are all down-sets. The set $\{b,d,e\}$ is not a down-set; we have $\downarrow\{b,d,e\} = \{a,b,c,d,e\}$. The set $\{e,f,g\}$ is an up-set, but $\{a,b,d,f\}$ is not.

(2) Figure 1.6 shows $\mathcal{O}(P)$ in a simple case.

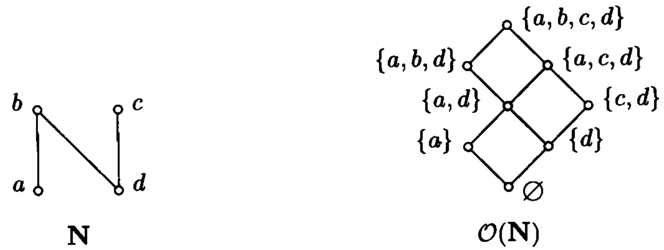


Figure 1.6

(3) If P is an antichain, then $\mathcal{O}(P) = \wp(P)$.

(4) If P is the chain $\mathbf{n}$, then $\mathcal{O}(P)$ consists of all the sets $\downarrow x$ for $x \in P$, together with the empty set. Hence $\mathcal{O}(P)$ is an $(n+1)$-element chain. If P is the chain of rational numbers, $\mathbb{Q}$, then $\mathcal{O}(P)$ contains the empty set, $\mathbb{Q}$ itself and all sets $\downarrow x$ (for $x \in \mathbb{Q}$). There are other sets in $\mathcal{O}(P)$ too: for example, $\downarrow x \setminus \{x\}$ (for $x \in \mathbb{Q}$) and $\{y \in \mathbb{Q} \mid y < a\}$ (for $a \in \mathbb{R} \setminus \mathbb{Q}$).

We conclude this section with a useful lemma connecting the order relation and down-sets. The proof is an easy but instructive exercise.

1.21 Lemma. *Let P be an ordered set and $x, y \in P$. Then the following are equivalent:*

(i) $x \leqslant y$;

(ii) $\downarrow x \subseteq \downarrow y$;

(iii) $(\forall Q \in \mathcal{O}(P))\, y \in Q \implies x \in Q$.

Maximal and minimal elements; top and bottom

We next introduce some important special elements.

1.22 Maximal and minimal elements. Let P be an ordered set and let $Q \subseteq P$. Then

(i) $a \in Q$ is a **maximal** element of Q if $a \leqslant x \in Q$ implies $a = x$;

(ii) $a \in Q$ is the **greatest** (or **maximum**) element of Q if $a \geqslant x$ for every $x \in Q$, and in that case we write $a = \max Q$.

Figure 1.7

A **minimal** element of Q, the **least** (or **minimum**) element of Q and $\min Q$ are defined dually, that is by reversing the order. Observe that Q has a greatest element only if it has precisely one maximal element, and that the greatest element of Q, if it exists, is unique (by the antisymmetry of $\leqslant$). Any non-empty subset of a finite ordered set P always has at least one maximal element. Figure 1.7 illustrates the definitions: P_1 has maximal elements a_1, a_2, a_3, but no greatest element; a is the greatest element of P_2.

A subset of the chain $\mathbb{N}$ has a maximal element if and only if it is finite. In the subset of $\wp(\mathbb{N})$ consisting of all subsets of $\mathbb{N}$ except $\mathbb{N}$ itself, there is no greatest element, but $\mathbb{N} \setminus \{n\}$ is maximal for each

$\langle P; \leqslant_1 \rangle$ is a chain and for all $a, b \in P$ we have $a \leqslant b$ implies $a \leqslant_1 b$. [The infinite case will be proved in Exercise 4.22.]

(iii) Use (i) to show that if $\langle P; \leqslant \rangle$ is a finite ordered set, then there is a finite number of chains $\langle P; \leqslant_1 \rangle, \ldots, \langle P; \leqslant_n \rangle$ such that for all $a, b \in P$ we have

$$a \leqslant b \iff (a \leqslant_1 b \,\&\ldots\&\, a \leqslant_n b).$$

(iv) Show that $\leqslant_1$ is a linear extension of the order $\leqslant$ on P if and only if $\langle P; \leqslant_1 \rangle$ is a chain and the identity map $\mathrm{id}\colon P \to P$ is an order-preserving map from $\langle P; \leqslant \rangle$ to $\langle P; \leqslant_1 \rangle$.

(v) Draw and label a diagram for every possible linear extension of the ordered set $\langle \mathbf{N}; \leqslant \rangle$ given in Figure 1.6.

1.23 Let P be a finite ordered set. The **width** of P is defined to be the size of the largest antichain in P and is denoted by $w(P)$.

(i) Find $w(P)$ for each of the ordered sets P in Figure 1.11 and show that in each case that P can be written as a union of $w(P)$ many chains.

(ii) Show that if a finite ordered set P can be written as the union of n chains, then $n \geqslant w(P)$.

(iii) **Dilworth's Theorem** states that the width $w(P)$ of a finite ordered set P equals the least $n \in \mathbb{N}$ such that P can be written as a union of n chains. The more intrepid may try to find their own proof of this important result. Alternatively, a much easier, but still valuable, exercise is to rewrite the snappy 14–line proof from [34], explaining every step in detail.

1.24 Show that it is impossible to find an order $\leqslant$ on $\mathbb{C}$ such that for all $w, z \in \mathbb{C}$

(a) $z = 0$ or $z > 0$ or $z < 0$,

(b) $w, z > 0$ imply $(-w) < 0$, $w + z > 0$ and $wz > 0$.

1.25 Let X be a topological space satisfying

(T_0): given $x \neq y$ in X there exists either an open set U such that $x \in U$, $y \notin U$ or an open set V such that $x \notin V$, $y \in V$.

Show that $\leqslant$, defined by $x \leqslant y$ if and only if $x \in \overline{\{y\}}$, is an order on X. Describe the down-sets and the up-sets for this order.

2

Lattices and Complete Lattices

Many important properties of an ordered set P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. Two of the most important classes of ordered sets defined in this way are lattices and complete lattices.

Lattices as ordered sets

It is a fundamental property of the real numbers, $\mathbb{R}$, that if I is a closed and bounded interval in $\mathbb{R}$, then every subset of I has both a least upper bound (or supremum) and a greatest lower bound (or infimum) in I. These concepts pertain to any ordered set.

2.1 Definitions. Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an **upper bound** of S if $s \leqslant x$ for all $s \in S$. A **lower bound** is defined dually. The set of all upper bounds of S is denoted by S^u (read as 'S **upper**') and the set of all lower bounds by S^ℓ (read as 'S **lower**'):

$$S^u := \{\, x \in P \mid (\forall s \in S)\, s \leqslant x \,\} \text{ and } S^\ell := \{\, x \in P \mid (\forall s \in S)\, s \geqslant x \,\}.$$

Since $\leqslant$ is transitive, S^u is always an up-set and S^ℓ a down-set. If S^u has a least element, x, then x is called the **least upper bound** of S. Equivalently, x is the least upper bound of S if

(i) x is an upper bound of S, and

(ii) $x \leqslant y$ for all upper bounds y of S.

Dually, if S^ℓ has a largest element, x, then x is called the **greatest lower bound** of S. Since least elements and greatest elements are unique (see 1.22), least upper bounds and greatest lower bounds are unique when they exist. The least upper bound of S is also called the **supremum** of S and is denoted by $\sup S$; the greatest lower bound of S is also called the **infimum** of S and is denoted by $\inf S$.

2.2 Remark. The two extreme cases, where S is empty or S is P itself, warrant a brief investigation. Recall from 1.23 that, when they exist, the top and bottom elements of P are denoted by $\top$ and $\bot$ respectively. It is easily seen that if P has a top element, then $P^u = \{\top\}$ in which case $\sup P = \top$. When P has no top element, we have $P^u = \varnothing$ and hence $\sup P$ does not exist. By duality, $\inf P = \bot$ whenever P has a bottom element. Now let S be the empty subset of P. Then every element

$x \in P$ satisfies (vacuously) $s \leqslant x$ for all $s \in S$. Thus $\varnothing^u = P$ and hence $\sup \varnothing$ exists if and only if P has a bottom element, and in that case $\sup \varnothing = \bot$. Dually, $\inf \varnothing = \top$ whenever P has a top element.

2.3 Notation. Looking ahead to Chapter 5, we shall adopt the following neater notation: we write $x \vee y$ (read as 'x **join** y') in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ (read as 'x **meet** y') in place of $\inf\{x, y\}$ when it exists. Similarly we write $\bigvee S$ (the '**join of** S') and $\bigwedge S$ (the '**meet of** S') instead of $\sup S$ and $\inf S$ when these exist. It is sometimes necessary to indicate that the join or meet is being found in a particular ordered set P, in which case we write $\bigvee_P S$ or $\bigwedge_P S$.

2.4 Remarks.

(1) Let P be any ordered set. If $x, y \in P$ and $x \leqslant y$, then $\{x, y\}^u = {\uparrow}y$ and $\{x, y\}^\ell = {\downarrow}x$. Since the least element of ${\uparrow}y$ is y and the greatest element of ${\downarrow}x$ is x, we have $x \vee y = y$ and $x \wedge y = x$ whenever $x \leqslant y$. In particular, since $\leqslant$ is reflexive, we have $x \vee x = x$ and $x \wedge x = x$.

(2) In an ordered set P, the least upper bound, $x \vee y$, of $\{x, y\}$ may fail to exist for two different reasons:

(a) because x and y have no common upper bound, or

(b) because they have no *least* upper bound.

In Figure 2.1(a) we have $\{a, b\}^u = \varnothing$ and hence $a \vee b$ doesn't exist. In (b) we find that $\{a, b\}^u = \{c, d\}$ and thus $a \vee b$ doesn't exist as $\{a, b\}^u$ has no least element.

(a) (b)

Figure 2.1

(3) Consider the ordered set drawn in Figure 2.2. It is tempting, at first sight, to think that $b \vee c = i$. On more careful inspection we find that $\{b, c\}^u = \{\top, h, i\}$. Since $\{b, c\}^u$ has distinct minimal elements, namely h and i, it cannot have a least element; hence $b \vee c$ does not exist. On the other hand, $\{a, b\}^u = \{\top, h, i, f\}$ has a least element, namely f, and thus $a \vee b = f$.

We shall be particularly interested in ordered sets in which $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$.

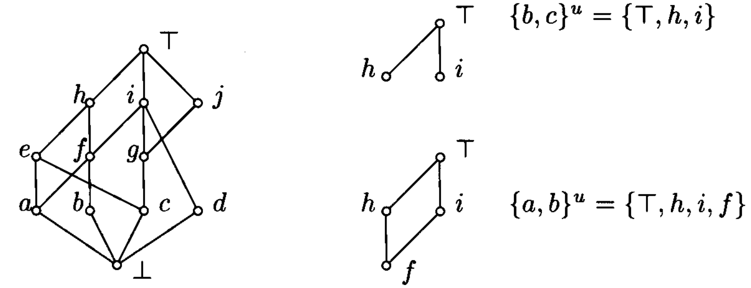


Figure 2.2

2.5 Definitions. Let P be a non-empty ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a **lattice**.

(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a **complete lattice**.

If P is a lattice, then $\vee$ and $\wedge$ are binary operations on P and we have an algebraic structure $\langle P; \vee, \wedge \rangle$. This theme will be developed in Chapter 5.

2.6 Remarks.

(1) In 2.5(ii), $S = \varnothing$ is allowed. Hence, by 2.2, any complete lattice is **bounded**, that is, has top and bottom elements.

(2) Let P be a non-empty ordered set. By Remark 2.4(1), if $x \leqslant y$ then $x \vee y = y$ and $x \wedge y = x$. Hence to show that P is a lattice it suffices to prove that $x \vee y$ and $x \wedge y$ exist in P for all non-comparable pairs $x, y \in P$.

2.7 Examples.

(1) By 2.6(2), every chain is a lattice in which $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Thus each of $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$ is a lattice under its usual order. None of them is complete; every one lacks a top element. However, if $-\infty < x < y < \infty$, then the closed interval $[x, y]$ in $\mathbb{R}$ is a complete lattice (by the completeness axiom for $\mathbb{R}$). Failure of completeness in $\mathbb{Q}$ is more fundamental than in $\mathbb{R}$. In $\mathbb{Q}$, it is not merely the lack of top and bottom elements which causes problems; for example, the set $\{\, s \in \mathbb{Q} \mid s^2 < 2 \,\}$ has upper bounds but no least upper bound in $\mathbb{Q}$.

(2) **The while-loop.** In the examples above we started from a fixpoint equation and sought its solution(s). More commonly in applications, the starting point is a recursive definition of some map or procedure, with the object to be defined recognizable as a solution of some fixpoint equation. To illustrate, we return to the programming language fragment we used in 3.52 to introduce denotational semantics. We ask how commands of the form '`while` B `do` C' should be assigned a meaning. Intuitively, the interpretation should be: 'so long as B is true, do C repeatedly; once B is false, stop in current state'. Thus in the notation of 3.52 we want, for any state σ, $\mathcal{C}[\![\texttt{while } B \texttt{ do } C]\!]\sigma$ to be $\mathcal{C}[\![\texttt{while } B \texttt{ do } C]\!]\mathcal{C}[\![C]\!]\sigma$ if $\mathcal{B}[\![B]\!]$ is true and σ otherwise. More succinctly, we are demanding

$$\mathcal{C}[\![\texttt{while } B \texttt{ do } C]\!] = cond(\mathcal{B}[\![B]\!], \mathcal{C}[\![\text{while } B \text{ do } C]\!] \circ \mathcal{C}[\![C]\!], \text{id}),$$

where *cond* is as defined in 3.52 and $\text{id}(\sigma) = \sigma$ for any state σ. This appears to require $\mathcal{C}[\![\texttt{while } B \texttt{ do } C]\!]$ to be defined in terms of itself. The position is clarified by writing A for $\mathcal{C}[\![\texttt{while } B \texttt{ do } C]\!]$ (a member of the domain $D := (St_\bot \to St_\bot)$) and Φ for the map from D to D which sends f to $cond(\mathcal{B}[\![B]\!], f \circ \mathcal{C}[\![C]\!], \text{id})$. Then the while-loop equation becomes $\Phi(A) = A$. Thus the problem of showing that the recursive procedure '`while` B `do` C' does have a proper definition is reduced to that of showing that a certain fixpoint equation has a solution.

We now begin to tackle the problem of the existence and construction of solutions to fixpoint equations, in an order-theoretic framework.

4.3 Technical note. Recall that we proved in 3.18 that for any $S \subseteq \mathbb{R}$ the set of maps from S to the flat CPO $S_\bot$ (or, equivalently, the set $(S \multimap S)$) is a CPO. Our preliminary examples therefore suggest that CPOs provide a suitable setting for our study. We need CPOs, not pre-CPOs, but (see the proof of 4.5) we do not require our maps to preserve $\bot$.

In the discussion of CPOs in Chapter 3 we needed to work with directed sets; here the emphasis is on chains. This allows us to by-pass directed sets and so reduce the dependence of this chapter on the last. We now take a CPO to be an ordered set with $\bot$ such that $\bigvee C$ (or, in the notation of 3.9, $\bigsqcup C$) exists in P for every non-empty chain C in P. The comments in 3.10 imply that this does not conflict with the definition in 3.9. Consequential amendments need to be made to other definitions, so that a continuous map becomes a map preserving joins of chains, etc.

4.4 Definitions. Let P be an ordered set and let $\Phi: P \to P$ be a map. We say $x \in P$ is a **fixpoint** of Φ if $\Phi(x) = x$. The set of all fixpoints of Φ is denoted by $\text{fix}(\Phi)$; it carries the induced order. The least element of $\text{fix}(\Phi)$, when it exists, is denoted by $\mu(\Phi)$.

The n-fold composite, Φ^n, of a map $\Phi : P \to P$ is defined as follows: Φ^n is the identity map if $n = 0$ and $\Phi^n = \Phi \circ \Phi^{n-1}$ for $n \geqslant 1$. If Φ is order-preserving, so is Φ^n.

Our first fixpoint theorem puts forward a candidate for the least fixpoint of an order-preserving map on a CPO, and confirms that the given construction always works when the map is continuous.

4.5 CPO Fixpoint Theorem I. *Let P be a CPO, let $\Phi: P \to P$ be an order-preserving map and define $\alpha := \bigsqcup_{n\geqslant 0} \Phi^n(\bot)$.*

(i) *If $\alpha \in \text{fix}(\Phi)$, then $\alpha = \mu(\Phi)$.*

(ii) *If Φ is continuous then the least fixpoint $\mu(\Phi)$ exists and equals α.*

Proof. (i) Certainly $\bot \leqslant \Phi(\bot)$. Applying the order-preserving map Φ^n, we have $\Phi^n(\bot) \leqslant \Phi^{n+1}(\bot)$, for all n. Hence we have a chain

$$\bot \leqslant \Phi(\bot) \leqslant \ldots \leqslant \Phi^n(\bot) \leqslant \Phi^{n+1}(\bot) \leqslant \ldots$$

in P. Since P is a CPO, $\alpha := \bigsqcup_{n\geqslant 0} \Phi^n(\bot)$ exists. Let β be any fixpoint of Φ. By induction, $\Phi^n(\beta) = \beta$ for all n. We have $\bot \leqslant \beta$, whence we obtain $\Phi^n(\bot) \leqslant \Phi^n(\beta) = \beta$ by applying Φ^n. The definition of α forces $\alpha \leqslant \beta$. Hence if α is a fixpoint then it is the least fixpoint.

(ii) It will be enough to show that $\alpha \in \text{fix}(\Phi)$. We have

$$\begin{aligned}
\Phi\Big(\bigsqcup_{n\geqslant 0} \Phi^n(\bot)\Big) &= \bigsqcup_{n\geqslant 0} \Phi(\Phi^n(\bot)) && \text{(since } \Phi \text{ is continuous)} \\
&= \bigsqcup_{n\geqslant 1} \Phi^n(\bot) \\
&= \bigsqcup_{n\geqslant 0} \Phi^n(\bot) && \text{(since } \bot \leqslant \Phi^n(\bot) \text{ for all } n\text{).}
\end{aligned}$$

∎

4.6 Remarks. An instructive parallel may be drawn between Theorem 4.5 and another well-known fixpoint theorem. Banach's Contraction Mapping Theorem asserts that a contraction map on a complete metric space has a fixpoint. It serves to show the existence of solutions to a variety of equations, in particular differential equations. Further, a fixpoint can be explicitly constructed by an iterative process analogous to that in 4.5. The approximating sequence converges because the metric space is complete, and its limit is a fixpoint thanks to the continuity of solution, namely α, the infinite string of alternating zeros and ones. This is exactly the solution we obtain by taking the empty string, $\varnothing$, and forming $\bigsqcup_{n\geqslant 1}\Phi^n(\varnothing)$. Clearly, $\Phi^n(\perp)$ is the $2n$-element string $0101\ldots01$. The join in Σ^{**} of these strings is α, which is certainly a fixpoint of Φ. Trivially Φ is order-preserving so, by Theorem 4.5(i), α is indeed the least fixpoint.

4.9 Recursively defined domains. Our examples so far in this chapter have concerned maps and procedures. We now hint at the potential of CPO Fixpoint Theorem I for creating solutions to domain equations of the kind we introduced in 3.52. As a simple example, consider the domain equation $\mathbf{F}(P)\cong P$, where $\mathbf{F}(P)=P_\perp$. Starting from $\mathbf{1}$ and forming $\{\mathbf{F}^n(\mathbf{1})\}_{n\geqslant 1}$, we obtain $\mathbf{1}_\perp$, $\mathbf{1}_\perp\cong\mathbf{2}$, $\mathbf{2}_\perp\cong\mathbf{3}$, etc. This makes it highly plausible that the least solution to $\mathbf{F}(P)\cong P$ is the domain $\mathbb{N}\oplus\mathbf{1}$. To confirm this, we must realize $\mathbf{F}$ as an order-preserving map on a CPO of domains and verify that $\mathbb{N}\oplus\mathbf{1}$ is indeed the join in this CPO of the approximations $\{\mathbf{F}^n(\mathbf{1})\}_{n\geqslant 0}$. We do not have an ordering of 'abstract' domains and must therefore realize domains concretely, either as $\bigcap$-structures (ordered by $\sqsubseteq$) or as information systems (ordered by $\trianglelefteq$). Using $\bigcap$-structures, we then have to construct $\mathbf{F}^n(\mathfrak{N})$. The reward for disentangling this notational horror (recall 3.46) is that the correct embeddings are then given at once via Exercise 3.29. This confirms that the 'natural' nesting of the chains $\mathbf{1},\mathbf{2},\ldots$ (in which each chain sits, in $\mathbb{N}\oplus\mathbf{1}$, as a down-set in the next) corresponds to the $\sqsubseteq$-ordering of their realizations as $\bigcap$-structures. See Figure 4.1 (in which $\mathbf{0}$ is used as an abbreviation for $(0,0)$).

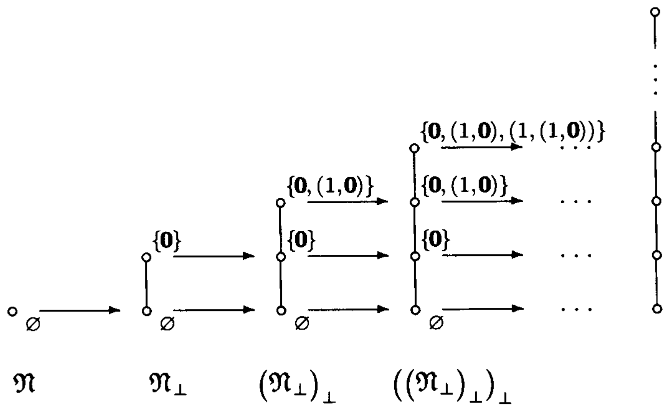


Figure 4.1

As a second example, consider the equation $P\cong P\oplus_\perp P$. Let $\mathbf{F}(P)=P\oplus_\perp P$. Then the least fixpoint of $\mathbf{F}$ is isomorphic to Σ^{**}, the domain of all binary strings. The nth approximation, $\mathbf{F}^n(\varnothing)$, to the solution is the set of strings of length at most n; again the ordering $\sqsubseteq$ is the 'natural' nesting. Finally we note that we cannot exhibit explicit solutions to domain equations, such as $P\cong(P\to P)_\perp$, which involve function spaces. As those who did Exercise 1.17 will realize, these spaces get very unwieldy very quickly and the limit domain cannot be visualized easily. However, so long as the domain constructor used is continuous, we know that there is a solution to the associated fixpoint equation, and this is what really matters, since it ensures that the models required for denotational semantics do indeed exist. Invoking CPO Fixpoint Theorem III below, we can even replace 'continuous' by 'order-preserving' above.

4.10 Remarks: the role of continuity. Consider the chain $P=\mathbb{N}\oplus\mathbf{2}$ and define Φ to be the (order-preserving) map fixing $\top$ and taking every other element to its upper cover. This map has $\top$ as its unique fixpoint, but $\top\neq\bigsqcup_{n\geqslant 0}\Phi^n(\perp)$. This example shows that for a non-continuous map Φ on a CPO we cannot expect $\bigsqcup\Phi^n(\perp)$ necessarily to provide a fixpoint.

Only for a map which is continuous does Theorem 4.5 guarantee the existence of a fixpoint. In simple cases it is possible to side-step the question of continuity by appealing to the first part of the theorem. This is convenient since verifying continuity can be a non-trivial undertaking, especially when the underlying CPO is a set of maps or the map Φ a domain constructor. It is much easier to decide whether a map preserves order, and we are led to ask whether an order-preserving map on a CPO must have a fixpoint. This turns out to be true though we have to work hard to establish it.

Before embarking on this proof we derive a classic fixpoint theorem due to Knaster and Tarski. This theorem is important in its own right and for the clues it provides to a systematic search for fixpoints more generally.

4.11 The Knaster–Tarski Fixpoint Theorem. *Let L be a complete lattice and $\Phi\colon L\to L$ an order-preserving map. Then*

$$\bigvee\{\,x\in L\mid x\leqslant\Phi(x)\,\}\in\operatorname{fix}(\Phi).$$

Proof. Let $H=\{\,x\in L\mid x\leqslant\Phi(x)\,\}$ and $\alpha=\bigvee H$. For all $x\in H$ we have $x\leqslant\alpha$, so $x\leqslant\Phi(x)\leqslant\Phi(\alpha)$. Thus $\Phi(\alpha)\in H^u$, whence $\alpha\leqslant\Phi(\alpha)$.

We now use this inequality to prove the reverse one (!) and thereby complete the proof that α is a fixpoint. Since Φ is order-preserving, $\Phi(\alpha) \leqslant \Phi(\Phi(\alpha))$. This says $\Phi(\alpha) \in H$, so $\Phi(\alpha) \leqslant \alpha$. ∎

4.12 Remark. The Knaster–Tarski Theorem shows that any order-preserving map on a powerset has a fixpoint. One such application yields as a by-product the famous Schröder–Bernstein Theorem stating that there is a bijection between sets A and B if there exist one-to-one maps from A to B and from B to A. For the proof, see Exercise 4.12. A full discussion of the Schröder-Bernstein Theorem and its uses in set theory can be found in [8].

The formula in Theorem 4.11 constructs what is easily seen to be the *greatest* fixpoint of Φ. A dual version of the proof produces $\mu(\Phi)$. Notice that both of the fixpoint theorems 4.5 and 4.11 find a fixpoint in the set $\{\, x \mid x \leqslant \Phi(x) \,\}$ of **pre-fixpoints**. The lemma below extracts the ideas underpinning 4.11.

4.13 Lemma. *Let P be an ordered set and let $\Phi\colon P \to P$ be a map. Let $H = \{\, x \in P \mid x \leqslant \Phi(x) \,\}$.*

(i) *Suppose $A \subseteq H$ and $\Phi(A) \subseteq A$. Then any maximal element of A belongs to* $\mathrm{fix}(\Phi)$.

(ii) *If Φ is order-preserving, then $\Phi(H) \subseteq H$ and*

  (a) $\bigvee_P H \in \mathrm{fix}(\Phi)$ *if this exists,*

  (b) *any maximal element of H belongs to* $\mathrm{fix}(\Phi)$.

Proof. (i) Assume α is a maximal point of A. Since $A \subseteq H$, we have $\alpha \leqslant \Phi(\alpha)$. We deduce that $\Phi(\alpha) = \alpha$, because $\Phi(\alpha) \in A$ and α is maximal in A.

(ii) The first two assertions are proved by the same techniques as were used in the proof of 4.11. For the last part take $A = H$ in (i). ∎

The next theorem is by far the hardest in this chapter, but yields substantial dividends later. It will enable us to prove easily that any order-preserving map on a CPO has a fixpoint and provides the key to our treatment of Zorn's Lemma in the next section.

4.14 CPO Fixpoint Theorem II. *Let P be a CPO and let $\Phi\colon P \to P$ be a map such that $x \leqslant \Phi(x)$ for all $x \in P$. Then Φ has a fixpoint.*

Proof. Lemma 4.13(ii) suggests a strategy for the proof. We seek a subset C of P such that $\Phi(C) \subseteq C$ (we shall call any such set **Φ-invariant**) and such that C has a maximal point. We know $\bigsqcup C$ exists in P for any non-empty chain. We therefore seek to choose C to be a non-empty chain which is Φ-invariant and which is a sub-CPO of P. Then the supremum of C actually lies *in* C (and thus is a maximal point of C). To build such a chain we might try to start from $\perp$ and add (towards achieving Φ-invariance) successively $\Phi(\perp)$, $\Phi(\Phi(\perp)), \ldots$. If this countable chain has a maximal element, we are home. Failing that, the chain does at least have a supremum γ and we may start climbing afresh, adding γ, $\Phi(\gamma)$, $\Phi(\Phi(\gamma)), \ldots$. The problem is to ensure that this process ever terminates. The formal proof starts from the other end, by taking a minimal Φ-invariant sub-CPO of P and showing that it is in fact a chain of the kind envisaged above.

Accordingly we let

$$\mathcal{E} := \{\, A \subseteq P \mid \Phi(A) \subseteq A \text{ and } A \text{ is a sub-CPO of } P \}$$

and define $C = \bigcap\{\, A \mid A \in \mathcal{E} \,\}$. Then C is a sub-CPO of P, by 3.12, and is easily seen to be Φ-invariant. If $A \in \mathcal{E}$ and $A \subseteq C$, then $A = C$. This observation is applied twice. On both occasions we take A to be those elements of C which satisfy some property we wish to establish for all of C. We then show that this set is in $\mathcal{E}$. (This is exactly the idea behind proofs by induction: a subset of $\mathbb{N}$ containing 1 and invariant under the successor function is the whole of $\mathbb{N}$.)

We wish to show C is a chain. This means

$$(\forall x \in C)((\forall y \in C)\, y \in {\downarrow} x \cup {\uparrow} x).$$

Because $x \leqslant \Phi(x)$ for all $x \in P$, this is certainly satisfied if C equals

$$A_x := \{\, y \in C \mid y \in {\downarrow} x \cup {\uparrow}\Phi(x) \,\}$$

for every $x \in C$. The strategy now is to define a set $A \subseteq C$ such $A \in \mathcal{E}$ and such that $x \in A$ implies that $A_x \in \mathcal{E}$. (Then we will be able to deduce that $A = C$ and thence that $A_x = C$ for all $x \in C$, as required.)

To see how to define A, we look at what is needed to ensure $A_x \in \mathcal{E}$. It is easy to check that A_x is a sub-CPO of P. What about Φ-invariance? Fix $x \in C$. To show that $y \in A_x$ implies $\Phi(y) \in A_x$ we require

(i) $y < x$ implies $\Phi(y) \leqslant x$ or $\Phi(y) \geqslant \Phi(x)$,

(ii) $y = x$ implies $\Phi(y) \leqslant x$ or $\Phi(y) \geqslant \Phi(x)$, and

(iii) $y \geqslant \Phi(x)$ implies $\Phi(y) \leqslant x$ or $\Phi(y) \geqslant \Phi(x)$.

Here (ii) is automatic and so is (iii), since $\Phi(y) \geqslant y$ for any $y \in P$; but (i) is not. We define A so that (i) is true if $x \in A$:

$$A = \{\, x \in C \mid (\forall y \in C)\,(y < x \Rightarrow \Phi(y) \leqslant x) \,\}.$$