Type Theory, Fall 2009

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Lecture: Formalising Mathematics in Type Theory

Types versus Sets

- Everything has a type

$$M{:}A$$

- Types are a bit like sets, but: ...
  - types give "syntactic information"

$$3 + (7 * 8)^5{:}\mathsf{nat}$$

  - sets give "semantic information"

$$3 \in \{n \in \mathbb{N} \mid \forall x, y, z > 0(x^n + y^n \neq z^n)\}$$

Per Martin-Löf:
A type comes with
construction principles: how to build objects of that type? and
elimination principles: what can you do with an object of that type?

This fits well with the Brouwerian view of mathematics:

> "there exists an x" means
> "we have a method of constructing x"

In short: a type is characterised by the construction principles for its objects.

Examples

- A natural number is either 0 or the successor S applied to a natural number.
  So the natural numbers are the objects of the shape $S(\ldots S(0)\ldots)$.
- A binary tree is either a leaf or the join of two binary trees.
- A proof of $n \leq m$ is either le_n, and then $m = n$, or le_S applied to a proof of $n \leq p$, and then $m = S(p)$.

Note:

> Checking whether an object belongs to an alleged type is decidable!

But if type checking should be decidable, there is not much information one can encode in a type (?)

$$X := \{n \in \mathbb{N} \mid \forall x, y, z > 0(x^n + y^n \neq z^n)\}$$

is X a type?
The proper question is: what are the objects of X? (How does one construct them?)

One constructs an object of the type X by giving an $N \in \mathbb{N}$ and a proof of the fact that $\forall x, y, z > 0(x^N + y^N \neq z^N)$.

The type X consists of pairs $\langle N, p \rangle$, with

- $N \in \mathbb{N}$
- p a proof of $\forall x, y, z > 0(x^N + y^N \neq z^N)$

$\langle N, p \rangle : X$ is decidable (if proof-checking is decidable).

More technically.
(Especially related to the type theory of Coq, but more widely applicable.)

- A data type (or set) is a term A : Set, or A : Type.
- A formula is a term φ : Prop
- An object is a term $t : A$ for some A : Set
- A proof is a term $p : \varphi$ for some φ : Prop.
- Set and Prop are both "universes" or "sorts".

Slogan: (Curry-Howard isomorphism)

Propositions	as	Types
Proofs	as	Terms

Judgement

$$\Gamma \vdash M : U$$

- Γ is a context
- M is a term
- U is a type

Two readings

- M is an object (expression) of data type U (if U : Set or U : Type)
- M is a proof (deduction) of proposition U (if U : Prop)

Γ contains

- variable declarations $x : T$
  - $x : A$ with A : Set/Type $\rightsquigarrow$ 'declaring x in A'
  - $x : \varphi$ with φ : Prop $\rightsquigarrow$ 'assuming φ' (axiom)
- definitions $x := M : T$
  - $x := t : A$ with A : Set/Type $\rightsquigarrow$ 'defining x as the expression t'
  - $x := p : \varphi$ with φ : Prop $\rightsquigarrow$ 'defining x as the proof p of φ'
    ($\simeq$ declaring x as a "reference" to the lemma φ)

Type theory as a basis for theorem proving

- Interactive theorem proving = interactive term construction
  Proving φ = (interactively) constructing a *proof term* $p : \varphi$
- Proof checking = Type checking
  Type checking is decidable and hence proof checking is.

Decidability problems:

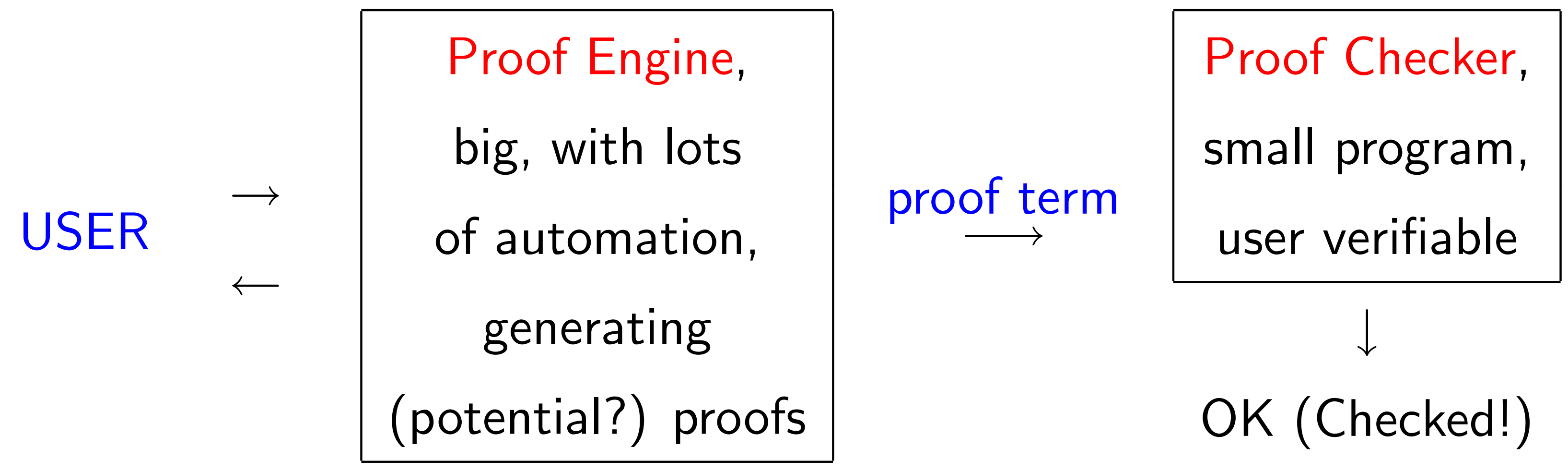
$\Gamma \vdash M : A?$	Type Checking Problem	TCP
$\Gamma \vdash M : ?$	Type Synthesis Problem	TSP
$\Gamma \vdash ? : A$	Type Inhabitation Problem	TIP

TCP and TSP are decidable
TIP is undecidable

De Bruijn criterion for theorem provers / proof checkers

How to check the checker?

Interactive Theorem Prover:



A TP satisfies the De Bruijn criterion if a small, 'easily' verifiable, independent proof checker can be written.

How proof terms occur (in Coq)

```
Lemma trivial : forall x:A, P x -> P x.
intros x H.
exact H.
Qed.
```

- Using the tactic script a term of type `forall x:A, P x -> P x` has been created.
- Using `Qed`, `trivial` is defined as this term and added to the global context.

Computation

- (β): $(\lambda x{:}A.M)N \to_\beta M[N/x]$
- (ι): primitive recursion reduction rules (inductive types)
- (δ): definition unfolding: if $x := t : A \in \Gamma$, then

$$M(x) \to_\delta^\Gamma M(t)$$

- Transitive, reflexive, symmetric closure: $=_{\beta\iota\delta}$

NB: Types that are equal modulo $=_{\beta\iota\delta}$ have the same inhabitants (definitional equality):

$$\text{(conversion)}\ \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma \vdash M : B}\ A =_{\beta\iota\delta} B$$

The Poincaré principle

Says that if $x : A(n) \to B$ and $y : A(f\,m)$, then

$$x\,y : B \text{ iff } f\,m = n \text{ by a computation.}$$

But: type checking should be decidable, so $f\,m = n$ should be decidable.

So: the definable functions in our type theory must be restricted: all computations should terminate.

Example of the recursion scheme (`1` abbreviates `(S 0)` etc.)

```
Fixpoint nfib (n:nat) :nat :=
match n with
 | 0     => 0
 | S m => match m with
             | 0     => 1
             | S p  =>  nfib p + nfib m
            end
end.
```

NB: Recursive calls should be 'smaller' (according to some rather general syntactic measure)

- Coq includes a (small, functional) programming language in which executable functions can be written.

Dependently typed data types: vectors of length n over A

```
Inductive vect (A:Set) : nat -> Set :=
    |  nnil   : vect A 0
    | ccons : forall (n:nat)(a:A), vect A n -> vect A (S n).
```

Now define, for example,

- head : forall (A:Set)(n:nat), vect A (S n) → A
- tail : forall (A:Set)(n:nat), vect A (S n) → vect A n

Let the type checker do the work for you!

Implicit Syntax

If the type checker can infer some arguments, we can leave them out:

Write $f \_ \_ a\, b$ in stead of $f\, S\, T\, a\, b$ if
$f : \Pi S, T{:}\mathsf{Set}. S \to T \to T$

Also: define $F := f \_ \_$ and write $F\, a\, b$.

Use Σ-types for mathematical structures

theory of groups: Given A : Type, a group over A is a tuple consisting of

$$\begin{array}{rcl} \circ & : & A{\to}A{\to}A \\ e & : & A \\ \mathsf{inv} & : & A{\to}A \end{array}$$

such that the following types are inhabited.

$$\begin{array}{rcl} \forall x,y,z{:}A.(x\circ y)\circ z & = & x\circ(y\circ z), \\ \forall x{:}A.e\circ x & = & x, \\ \forall x{:}A.(\mathsf{inv}\ x)\circ x & = & e. \end{array}$$

Type of group-structures over A, Group-Str(A), is

$$(A{\to}A{\to}A)\times(A\times(A{\to}A))$$

The type of groups over A, Group(A), is

$$\text{Group}(A) := \Sigma \circ{:}A{\rightarrow}A{\rightarrow}A.\Sigma e{:}A.\Sigma \text{inv}{:}A{\rightarrow}A.$$
$$(\forall x, y, z{:}A.(x \circ y) \circ z = x \circ (y \circ z))\wedge$$
$$(\forall x{:}A.e \circ x = x)\wedge$$
$$(\forall x{:}A.(\text{inv } x) \circ x = e).$$

If t : Group(A), we can extract the elements of the group structure by projections: $\pi_1 t : A{\rightarrow}A{\rightarrow}A$, $\pi_1(\pi_2 t) : A$

If $f : A{\rightarrow}A{\rightarrow}A$, $a : A$ and $h : A{\rightarrow}A$ with p_1, p_2 and p_3 proof-terms of the associated group-axioms, then

$$\langle f, \langle a, \langle h, \langle p_1, \langle p_2, p_3\rangle\rangle\rangle\rangle\rangle : \text{Group}(A).$$

We would like to use names for the projections:
Coq has labelled record types (type dependent)

-
```
Record My_type : Set :=
  { l_1   : type_1 ;
    l_2   : type_2 ;
    l_3   : type_3   }.
```
  If `X : My_type`, then `(l_1 X) : type_1`.

- Basically, `My_type` consists of labelled tuples:
  `[l_1:= value_1, l_2:=value_2, l_3:=value_3]`

- Also with dependent types: `l_1` may occur in `type_2`.
  If `X : My_type`, then

```
(l_2 X) : type_2 [(l_1 X)/l_1]
```

- ```
  Record Group : Type :=
    { cr   :  Set;
      op    :  cr -> cr -> cr;
      unit  :  cr;
      inv   :  cr -> cr;
      assoc :  forall x y z : cr,
               op (op x y) z = op x (op y z)
      ...      ...
     }.
  ```

  If `X : Group`, then `(op X) : (cr X) -> (cr X) -> (cr X)`.

The record types can be defined in Coq using inductive types.

Note: `Group` is in `Type` and not in `Set`
  ```

Coercions

- The user can tell the type checker to use specific terms as coercions.
  `Coercion k : A >-> B` declares the term `k : A -> B` as a coercion.
  - If `f a` can not be typed, the type checker will try to type check `(k f) a` and `f (k a)`.
  - If we declare a variable `x:A` and `A` is not a type, the type checker will check if `(k A)` is a type.

  Coercions can be composed.

```
Record Monoid : Type :=
  { m_cr    :> Semi_grp;
    m_proof :  (Commutative m_cr (sg_op m_cr))
        /\ (IsUnit m_cr (sg_unit m_cr) (sg_op m_cr)) }.
```

- A monoid is now a tuple $\langle\langle\langle S, =_S, r\rangle, a, f, p\rangle, q\rangle$

  If `M : Monoid`, the carrier of `M` is `(cr(sg_cr(m_cr M)))`

  Nasty !!

  ⇒ We want to use the structure `M` as synonym for the carrier set `(cr(sg_cr(m_cr M)))`.

  ⇒ The maps `cr`, `sg_cr`, `m_cr` should be left implicit.

- The notation `m_cr :> Semi_grp` declares the coercion `m_cr : Monoid >-> Semi_grp`.

Inheritance via Coercions

We have the following coercions.

```
OrdField >-> Field >-> Ring >-> Group

Group >-> Monoid >-> Semi_grp >-> Setoid
```

- All properties of groups are inherited by rings, fields, etc.
- Also notation can be inherited: `x[+]y` denotes the addition of `x` and `y` for `x,y:G` from any semi-group (or monoid, group, ring,...) `G`.
- The coercions must form a tree, so there is no real *multiple inheritance*:
  E.g. it is *not* possible to define rings in such a way that it inherits both from its additive group and its multiplicative monoid.

- Set theory (and logic): a function $f : A \to B$ is a relation $R \subset A \times B$ such that $\forall x{:}A.\exists! y{:}B.R\,x\,y$. "functions as graphs"
- In Type theory, we have functions-as-graphs ($R : A \to B \to \mathsf{Prop}$), but also functions-as-algorithms: $f : A \to B$.

Functions as algorithms also compute: β and ι rules:

$$\begin{array}{rcl}(\lambda x{:}A.M)N & \longrightarrow_\beta & M[N/x], \\ \mathsf{Rec}\, b\, f\, 0 & \longrightarrow_\iota & b, \\ \mathsf{Rec}\, b\, f\, (S\, x) & \longrightarrow_\iota & f\, x\, (\mathsf{Rec}\, b\, f\, x).\end{array}$$

Terms of type $A \to B$ denote algorithms, whose operational semantics is given by the reduction rules.
(Type theory as a small programming language)

Intensionality versus Extensionality

The equality in the side condition in the (conversion) rule can be intensional or extensional.

Extensional equality requires the following rules:

$$\text{(ext)} \quad \frac{\Gamma \vdash M, N : A \to B \quad \Gamma \vdash p : \Pi x{:}A.(Mx = Nx)}{\Gamma \vdash M = N : A \to B}$$

$$\text{(conv)} \quad \frac{\Gamma \vdash P : A \quad \Gamma \vdash A = B : s}{\Gamma \vdash P : B}$$

- Intensional equality of functions = equality of algorithms (the way the function is presented to us (syntax))
- Extensional equality of functions = equality of graphs (the (set-theoretic) meaning of the function (semantics))

Adding the rule (ext) renders TCP undecidable

Suppose $H : (A \rightarrow B) \rightarrow \text{Prop}$ and $x : (H\ f)$; then

$$x : (H\ g) \text{ iff there is a } p : \Pi x{:}A. f\ x = g\ x$$

So, to solve this TCP, we need to solve a TIP.

The interactive theorem prover Nuprl is based on extensional type theory.

"An equality involving a computation does not require a proof"

In type theory: if $t = q$ by evaluation (computing an algorithm), then this is a trivial equality, proved by reflexivity.
This is made precise by the conversion rule:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash M : B}\ A =_{\beta\iota\delta} B$$

Can we actually use the programming power of Type Theory when formalizing mathematics?

Yes. For automation: replacing a proof obligation by a computation

- Suppose we have a class of problems with a syntactic encoding as a data type, say via the type Problem.
  Example: equalities between expressions over a group

```
Inductive E : Set :=
   evar  : nat -> E
 | eone  : E
 | eop   : E -> E -> E
 | einv  : E -> E
```

- Suppose we have a decoding function $[\![-]\!]$: Problem $\rightarrow$ Prop
- Suppose we have a decision function Dec : Problem $\rightarrow \{0, 1\}$
- Suppose we can prove Ok : $\forall p$:Problem((Dec(p) $= 1) \rightarrow [\![p]\!])$

To verify P (from the class of problems):

- Find a p : Problem such that $[\![p]\!] = P$.
- Then Dec(p) yields either 1 or 0
- If Dec$(p) = 1$, then we have a proof of P (using Ok)
- If Dec$(p) = 0$, we obtain no information about P (it 'fails')

Note: if Dec is complete:

$$\forall p{:}\text{Problem}((\text{Dec}(p) = 1) \leftrightarrow [\![p]\!])$$

then Dec$(p) = 0$ yields a proof of $\neg P$.

How to represent the notion of set?

Note: A set is not just a type, because

$M : A$ is decidable whereas $t \in X$ is undecidable

A setoid is a pair $[A, =]$ with

- A : Set,
- $= : A \to (A \to \text{Prop})$ an equivalence relation over A

Function space setoid (the setoid of setoid functions)

$[A \stackrel{s}{\to} B, =_{A \stackrel{s}{\to} B}]$ is defined by

$$\begin{aligned} A \stackrel{s}{\to} B \quad &:= \quad \Sigma f{:}A \to B.(\Pi x, y{:}A.(x =_A y) \to ((f\ x) =_B\ (f\ y))), \\ f =_{A \stackrel{s}{\to} B} g \quad &:= \quad \Pi x, y{:}A.(x =_A y) \to (\pi_1\ f\ x) =_B (\pi_1\ g\ y). \end{aligned}$$

Q is an equivalence relation over the setoid $[A, =_A]$ if

- $Q : A{\to}(A{\to}\mathsf{Prop})$ is an equivalence relation,
- $=_A \subset Q$, i.e. $\forall x, y{:}A.(x =_A y){\to}(Q\ x\ y)$.

The quotient setoid $[A, =_A]/Q$ is defined as

$$[A, Q]$$

Easy exercise:
If the setoid function $f : [A, =_A] \to [B, =_B]$ respects Q
(i.e. $\forall x, y{:}A.(Q\ x\ y){\to}((f\ x) =_B (f\ y))$)
it induces a setoid function from $[A, =_A]/Q$ to $[B, =_B]$.

Given $[A, =_A]$ and predicate P on A define the sub-setoid

$$[A, =_A] \mid P \;\; := \;\; [\Sigma x{:}A.(P\ x), =_A|P]$$

$=_A|P$ is $=_A$ restricted to P: for $q, r : \Sigma x{:}A.(P\ x)$,

$$q\ (=_A|P)\ r \quad := \quad (\pi_1\ q) =_A (\pi_1\ r)$$

Proof-irrelevance is "embedded" in the subsetoid construction:

Setoid functions are proof-irrelevant.

Objects depending on proofs

What should be the type of the reciprocal?

- Let recip : $A \to A$ with $\forall x{:}A.x \neq 0 \to \mathsf{mult}\, x(\mathsf{recip}\, x) = 1$
- Either leave recip 0 unspecified (Mizar) or make an arbitrary choice for it (HOL). But it should be undefined
- Type theoretic solution

$$\mathsf{recip} : (\Sigma x{:}A.x \neq 0) \to A$$

- Then recip is only defined on elements that are non-zero: recip takes as input a pair $\langle a, p\rangle$ with $p : a \neq 0$ and returns recip$\langle a, p\rangle : A$.
- How to understand the dependency of this object (of type A) on the proof p?

- Take a setoid $[A, =_A]$ as the carrier of a field
- The operations on the field are taken to be setoid functions
- The field-properties are now denoted using the setoid equality.
  For the reciprocal:

$$\text{recip} : [A, =_A] \mid (\lambda x{:}A.x \neq_A 0) \to [A, =_A],$$

  a setoid function from the subsetoid of non-zeros to $[A, =_A]$

Note recip still takes a pair of an object and a proof $\langle a, p\rangle$ and returns recip$\langle a, p\rangle : A$.
But recip is now a setoid function which implies

$$\text{If } p : a \neq_A 0, q : a \neq_A 0, \text{ then } \text{recip}\langle a, p\rangle =_A \text{recip}\langle a, q\rangle$$

Summarizing (Proof terms in intensional type theory)

- The 'subtype' $\{t : A \mid (P\ t)\}$ is defined as the type of pairs $\langle t, p\rangle$ where $t : A$ and $p : (P\ t)$.
- Equality on this subtype is "just" equality on A.
- A partial function is a function on a subtype
  E.g. $(-)^{-1} : \{x{:}\mathbb{R} \mid x \neq 0\} \to \mathbb{R}$.
  If $x : \mathbb{R}$ and $p : x \neq 0$, then $\frac{1}{\langle x,p\rangle} : \mathbb{R}$.
- We only consider partial functions that are proof-irrelevant, i.e.
  if $p : t \neq 0$ and $q : t \neq 0$, then $\frac{1}{\langle t,p\rangle} = \frac{1}{\langle t,q\rangle}$.