Theorems For Free!

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Assume $a : A \to A'$ and $b : B \to B'$.

$$head : \forall X.\ X^* \to X$$
$$a \circ head_A = head_{A'} \circ a^*$$

$$tail : \forall X.\ X^* \to X^*$$
$$a^* \circ tail_A = tail_{A'} \circ a^*$$

$$(+\!\!+) : \forall X.\ X^* \to X^* \to X^*$$
$$a^*\ (xs +\!\!+_A ys) = (a^*\ xs) +\!\!+_{A'} (a^*\ ys)$$

$$concat : \forall X.\ X^{**} \to X^*$$
$$a^* \circ concat_A = concat_{A'} \circ a^{**}$$

$$fst : \forall X.\ \forall Y.\ X \times Y \to X$$
$$a \circ fst_{AB} = fst_{A'B'} \circ (a \times b)$$

$$snd : \forall X.\ \forall Y.\ X \times Y \to Y$$
$$b \circ snd_{AB} = snd_{A'B'} \circ (a \times b)$$

$$zip : \forall X.\ \forall Y.\ (X^* \times Y^*) \to (X \times Y)^*$$
$$(a \times b)^* \circ zip_{AB} = zip_{A'B'} \circ (a^* \times b^*)$$

$$filter : \forall X.\ (X \to Bool) \to X^* \to X^*$$
$$a^* \circ filter_A\ (p' \circ a) = filter_{A'}\ p' \circ a^*$$

$$sort : \forall X.\ (X \to X \to Bool) \to X^* \to X^*$$
$$\text{if for all } x, y \in A,\quad (x < y) = (a\ x <'\ a\ y) \text{ then}$$
$$a^* \circ sort_A\ (<) = sort_{A'}\ (<') \circ a^*$$

$$fold : \forall X.\ \forall Y.\ (X \to Y \to Y) \to Y \to X^* \to Y$$
$$\text{if for all } x \in A, y \in B,\quad b\ (x \oplus y) = (a\ x) \otimes (b\ y) \text{ and } b\ u = u' \text{ then}$$
$$b \circ fold_{AB}\ (\oplus)\ u = fold_{A'B'}\ (\otimes)\ u' \circ a^*$$

$$I : \forall X.\ X \to X$$
$$a \circ I_A = I_{A'} \circ a$$

$$K : \forall X.\ \forall Y.\ X \to Y \to X$$
$$a\ (K_{AB}\ x\ y) = K_{A'B'}\ (a\ x)\ (b\ y)$$

For any relations $\mathcal{A} : A \Leftrightarrow A'$ and $\mathcal{B} : B \Leftrightarrow B'$, the relation $\mathcal{A} \times \mathcal{B} : (A \times B) \Leftrightarrow (A' \times B')$ is defined by

$$((x, y), (x', y')) \in \mathcal{A} \times \mathcal{B}$$
$$\text{iff}$$
$$(x, x') \in \mathcal{A} \text{ and } (y, y') \in \mathcal{B}.$$

For any relation $\mathcal{A} : A \Leftrightarrow A'$, the relation $\mathcal{A}^* : A^* \Leftrightarrow A'^*$ is defined by

$$([x_1, \ldots, x_n], [x'_1, \ldots, x'_n]) \in \mathcal{A}^*$$
$$\text{iff}$$
$$(x_1, x'_1) \in a \text{ and } \ldots \text{ and } (x_n, x'_n) \in \mathcal{A}.$$

For any relations $\mathcal{A} : A \Leftrightarrow A'$ and $\mathcal{B} : B \Leftrightarrow B'$, the relation $\mathcal{A} \to \mathcal{B} : (A \to B) \Leftrightarrow (A' \to B')$ is defined by

$$(f, f') \in \mathcal{A} \to \mathcal{B}$$
$$\text{iff}$$
$$\text{for all } (x, x') \in \mathcal{A}, \quad (f\ x, f'\ x') \in \mathcal{B}.$$

$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
$$\text{iff}$$
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

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$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
iff
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

Tail

$$r : \forall X.\ X^* \to X^*.$$

$$(r, r) \in \forall \mathcal{X}.\ \mathcal{X}^* \to \mathcal{X}^*.$$

for all $\mathcal{A} : A \Leftrightarrow A'$,
$(r_A, r_{A'}) \in \mathcal{A}^* \to \mathcal{A}^*$

for all $\mathcal{A} : A \Leftrightarrow A'$,
for all $(xs, xs') \in \mathcal{A}^*$,
$(r_A\ xs, r_{A'}\ xs') \in \mathcal{A}^*$

for all $a : A \to A'$,
for all xs,
$a^*\ xs = xs'$ implies $a^*\ (r_A\ xs) = r_{A'}\ xs'$

for all $a : A \to A'$,
$a^* \circ r_A = r'_A \circ a^*$.

For any relations $\mathcal{A} : A \Leftrightarrow A'$ and $\mathcal{B} : B \Leftrightarrow B'$, the relation $\mathcal{A} \times \mathcal{B} : (A \times B) \Leftrightarrow (A' \times B')$ is defined by

$$((x, y), (x', y')) \in \mathcal{A} \times \mathcal{B}$$
$$\text{iff}$$
$$(x, x') \in \mathcal{A} \text{ and } (y, y') \in \mathcal{B}.$$

For any relation $\mathcal{A} : A \Leftrightarrow A'$, the relation $\mathcal{A}^* : A^* \Leftrightarrow A'^*$ is defined by

$$([x_1, \ldots, x_n], [x'_1, \ldots, x'_n]) \in \mathcal{A}^*$$
$$\text{iff}$$
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$$(f, f') \in \mathcal{A} \to \mathcal{B}$$
$$\text{iff}$$
$$\text{for all } (x, x') \in \mathcal{A}, \quad (f\ x, f'\ x') \in \mathcal{B}.$$

$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
$$\text{iff}$$
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

Fold

$$fold : \forall X.\ \forall Y.\ (X \to Y \to Y) \to Y \to X^* \to Y.$$

$$(fold, fold) \in \forall \mathcal{X}.\ \forall \mathcal{Y}.\ (\mathcal{X} \to \mathcal{Y} \to \mathcal{Y}) \to \mathcal{Y} \to \mathcal{X}^* \to Y.$$

$$(fold_{AB}, fold_{A'B'}) \in (a \to b \to b) \to b \to a^* \to b$$

for all $(\oplus, \oplus') \in (a \to b \to b)$,
  for all $(u, u') \in b$,
    $(fold_{AB}\ (\oplus)\ u, fold_{A'B'}\ (\oplus')\ u') \in a^* \to b.$

the condition $(\oplus, \oplus') \in (a \to b \to b)$ is equivalent to

for all $x \in A, x' \in A', y \in B, y' \in B'$,
  $a\ x = x'$ and $b\ y = y'$ implies $b\ (x \oplus y) = x' \oplus' y'$.

for all $a : A \to A', b : B \to B'$,
  if for all $x \in A, y \in B$, $\quad b\ (x \oplus y) = (a\ x) \oplus' (b\ y)$,
  and $b\ u = u'$
  then $b \circ fold_{AB}\ (\oplus)\ u = fold_{A'B'}\ (\oplus')\ u' \circ a^*$.

For any relations $\mathcal{A} : A \Leftrightarrow A'$ and $\mathcal{B} : B \Leftrightarrow B'$, the relation $\mathcal{A} \times \mathcal{B} : (A \times B) \Leftrightarrow (A' \times B')$ is defined by

$$((x, y), (x', y')) \in \mathcal{A} \times \mathcal{B}$$
iff
$$(x, x') \in \mathcal{A} \text{ and } (y, y') \in \mathcal{B}.$$

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$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
iff
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

Sort

$$s : \forall X.(X \to X \to Bool) \to (X^* \to X^*)$$

$$\text{if for all } x, y \in A, \quad (x \prec y) = (a\ x \prec'\ a\ y) \text{ then}$$
$$a^* \circ s_A(\prec) = s_{A'}(\prec') \circ a^*$$

$$\text{if for all } x, y \in A, \quad (x < y) = (a\ x <'\ a\ y) \text{ then}$$
$$sort_{A'}\ (<) \circ a^* = a^* \circ sort_A\ (<')$$

$$\text{if for all } x, y \in A, \quad (x \equiv y) = (a\ x \equiv'\ a\ y) \text{ then}$$
$$nub_{A'}\ (\equiv) \circ a^* = a^* \circ nub_A\ (\equiv')$$

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$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
iff
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

Polymorphic equality

from

$$(=) : \forall X.\ X \to X \to Bool.$$

We can derive:

$$\text{for all } x, y \in A, \quad (x =_A y) = (a\ x =_{A'} a\ y).$$

Which seems weird, because this function exists.
However, instead we can define:

$$(=) : \forall^{(=)} X.\ X \to X \to Bool.$$

Which corresponds with Miranda’s eqtypes or Haskells type classes

$$(g, g') \in \forall^{(=)} \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
iff
$$\text{for all } \mathcal{A} : A \Leftrightarrow A' \text{ respecting } (=), \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

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iff
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$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
iff
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

A result about map

Intuitively, every function that has the same type as map, can be composed of map and a reorder function

$$m : \forall X. \forall Y. (X \to Y) \to (X^* \to Y^*)$$

$$m_{AB}(f) = f^* \circ m_{AA}(I_A) = m_{BB}(I_B) \circ f^*$$

$$\text{if } f' \circ a = b \circ f \text{ then } m_{A'B'}(f') \circ a^* = b^* \circ m_{AB}(f)$$

$$m_{BB}(I_B) \circ f^* = (I_B)^* \circ m_{AB}(f)$$

For any relations $\mathcal{A} : A \Leftrightarrow A'$ and $\mathcal{B} : B \Leftrightarrow B'$, the relation $\mathcal{A} \times \mathcal{B} : (A \times B) \Leftrightarrow (A' \times B')$ is defined by

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For any relations $\mathcal{A} : A \Leftrightarrow A'$ and $\mathcal{B} : B \Leftrightarrow B'$, the relation $\mathcal{A} \to \mathcal{B} : (A \to B) \Leftrightarrow (A' \to B')$ is defined by

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$$\text{for all } (x, x') \in \mathcal{A}, \quad (f\ x, f'\ x') \in \mathcal{B}.$$

$$(g, g') \in \forall \mathcal{X}.\ \mathcal{F}(\mathcal{X})$$
iff
$$\text{for all } \mathcal{A} : A \Leftrightarrow A', \quad (g_A, g'_{A'}) \in \mathcal{F}(\mathcal{A}).$$

A result about fold

Intuitively, every function that has the same type as fold, can be composed of fold and a reorder function

$$f : \forall X . \forall Y . (X \to Y \to Y) \to Y \to X^* \to Y$$

$$f_{AB}\ c\ n = fold_{AB}\ c\ n \circ f_{AA^*}\ cons_A\ nil_A$$

if $c' \circ (a \times b) = b \circ c$ and $n' = b(n)$ then
$$f_{A'B'}\ c'\ n' \circ a^* = b \circ f_{AB}\ c\ n$$

$$f_{AB'}\ c'\ n' \circ I_A^* = fold_{AB'}\ c'\ n' \circ f_{AA^*}\ cons_A\ nil_A$$