Transform Target Languages

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Definitions

Call by value (CBV) strategy:

$$\begin{aligned} V &::= x \mid \lambda x : A.M \mid \Lambda u : K.M \\ R &::= (\lambda x : A.M)V \mid (\Lambda u : K.M)\{A\} \mid \\ &\quad\ \ abort_A(M) \mid callcc_A(M) \\ E &::= [\,] \mid EM \mid VE \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M)V] &\hookrightarrow_{cbv} E[[V/x]M] \\ E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{cbv} E[[A/u]M] \\ E[abort_A(M)] &\hookrightarrow_{cbv} M \\ E[callcc_A(M)] &\hookrightarrow_{cbv} E[M(\Lambda u : \Omega.\lambda x : A.abort_u(E[x]))] \\ & \qquad (u \notin FTV(A)) \end{aligned}$$

Definitions

Call by name (CBN) strategy:

$$
\begin{aligned}
V &::= \lambda x : A.M \mid \Lambda u : K.M \\
R &::= (\lambda x : A.M_1)M_2 \mid (\Lambda u : K.M)\{A\} \mid \\
&\qquad abort_A(M) \mid callcc_A(M) \\
E &::= [\,] \mid EM \mid E\{A\}
\end{aligned}
$$

$$
\begin{aligned}
E[(\lambda x : A.M_1)M_2] &\hookrightarrow_{cbn} E[[M_2/x]M_1] \\
E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{cbn} E[[A/u]M] \\
E[abort_A(M)] &\hookrightarrow_{cbn} M \\
E[callcc_A(M)] &\hookrightarrow_{cbn} E[M(\Lambda u : \Omega.\lambda x : A.abort_u(E[x]))] \\
&\qquad (u \notin FTV(A))
\end{aligned}
$$

Definitions

ML-CBV:

$$
\begin{aligned}
V &::= \; x \mid \lambda x : A.M \mid \Lambda u : K.M \\
R &::= \; (\lambda x : A.M)V \mid (\Lambda u : K.M)\{A\} \mid \\
&\qquad abort_A(M) \mid callcc_A(M) \\
E &::= \; [\,] \mid EM \mid VE \mid \Lambda u : K.E \mid E\{A\}
\end{aligned}
$$

$$
\begin{aligned}
E[(\lambda x : A.M)V] &\hookrightarrow_{ml-cbv} \; E[[V/x]M] \\
E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{ml-cbv} \; E[[A/u]M] \\
E[abort_A(M)] &\hookrightarrow_{ml-cbv} \; M \\
E[callcc_A(M)] &\hookrightarrow_{ml-cbv} \; E[M(\Lambda u : \Omega.\lambda x : A.abort_u(E[x]))] \\
&\qquad (u \notin FTV(A))
\end{aligned}
$$

Definitions

ML-CBN:

$$
\begin{aligned}
V &::= \lambda x : A.M \mid \Lambda u : K.M \\
R &::= (\lambda x : A.M_1)M_2 \mid (\Lambda u : K.M)\{A\} \mid \\
&\qquad abort_A(M) \mid callcc_A(M) \\
E &::= [\,] \mid EM \mid \Lambda u : K.E \mid E\{A\}
\end{aligned}
$$

$$
\begin{aligned}
E[(\lambda x : A.M_1)M_2] &\hookrightarrow_{ml-cbn} E[[M_2/x]M_1] \\
E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{ml-cbn} E[[A/u]M] \\
E[abort_A(M)] &\hookrightarrow_{ml-cbn} M \\
E[callcc_A(M)] &\hookrightarrow_{ml-cbn} E[M(\Lambda u : \Omega.\lambda x : A.abort_u(E[x]))] \\
&\qquad (u \notin FTV(A))
\end{aligned}
$$

Definitions

We say that two strategies *coincide* if the same β-reductions occur regardless of which strategy is used.

Standard CPS form

Standard CPS Form

Standard CPS values	$W ::= x \mid \lambda x : A.N \mid \Lambda u : K.N$
Standard CPS terms	$N ::= W \mid NW \mid N\{A\}$

Lemma 4.1

If N is a standard CPS term, then $[W_1/x]N$ is also a standard CPS term.

If W_2 is a standard CPS value, then $[W_2/x]W_1$ is also a standard CPS value.

standard CPS form properties (Theorem 4.2)

The standard CPS form has the following properties:

1. Standard CPS form is closed under CBV and CBN reductions.
2. If N_1 is a standard CPS term, then $N_1 \hookrightarrow_{cbv} N_2$ iff $N_1 \hookrightarrow_{cbn} N_2$.
3. CBV or CBN evaluation of well-typed, closed standard CPS terms terminates in a standard CPS value.

Proof

Looking at the definitions of CBV, CBN and the standard CPS form gives us the following evaluation strategy:

$$\begin{aligned} V &::= \quad W \\ R &::= \quad (\lambda x : A.N)V \mid (\Lambda u : K.N)\{A\} \\ E &::= \quad [\,] \mid EV \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.N)V] &\quad \hookrightarrow \quad E[[V/x]N] \\ E[(\Lambda u : K.N)\{A\}] &\quad \hookrightarrow \quad E[[A/u]N] \end{aligned}$$

that is equivalent to both CBV and CBN on standard CPS terms.

Proof

Where does this strategy come from?

Recall (a part of) the definition of the CBV strategy:

$$\begin{aligned} V &::= x \mid \lambda x : A.M \mid \Lambda u : K.M \\ R &::= (\lambda x : A.M)V \mid (\Lambda u : K.M)\{A\} \\ E &::= [] \mid EM \mid VE \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M)V] &\quad \hookrightarrow_{cbv} \quad E[[V/x]M] \\ E[(\Lambda u : K.M)\{A\}] &\quad \hookrightarrow_{cbv} \quad E[[A/u]M] \end{aligned}$$

Proof

And (a part of) the definition of the CBN strategy:

$$\begin{aligned} V &::= \lambda x : A.M \mid \Lambda u : K.M \\ R &::= (\lambda x : A.M_1)M_2 \mid (\Lambda u : K.M)\{A\} \\ E &::= [] \mid EM \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M_1)M_2] &\hookrightarrow_{cbn} E[[M_2/x]M_1] \\ E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{cbn} E[[A/u]M] \end{aligned}$$

Proof

Compare the reductions of CBV and CBN:

$$\begin{aligned} E[(\lambda x : A.M)V] &\quad \hookrightarrow_{cbv} \quad E[[V/x]M] \\ E[(\Lambda u : K.M)\{A\}] &\quad \hookrightarrow_{cbv} \quad E[[A/u]M] \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M_1)M_2] &\quad \hookrightarrow_{cbn} \quad E[[M_2/x]M_1] \\ E[(\Lambda u : K.M)\{A\}] &\quad \hookrightarrow_{cbn} \quad E[[A/u]M] \end{aligned}$$

Proof

These strategies are equivalent:

CBV and CBN coincide for Λ.

Looking at standard CPS terms ($N ::= W \mid NW \mid N\{A\}$) shows us that the argument of a function application is a standard CPS value.

So CBV and CBN also coincide for λ.

ML-CPS form

Why ML-CPS form?

ML-CBV and ML-CBN do not coincide on standard CPS terms.

Example:

$$(\lambda x : (\forall u : K.A).x)(\Lambda u : K.(\lambda y : A.y)c)$$

ML-CBV will do the innermost redex first, ML-CBN will do the outermost redex first.

ML-CPS form definition

ML-CPS values	$X ::= x \mid \lambda x : A.O \mid \Lambda u : K.X$
ML-CPS terms	$O ::= X \mid OX \mid \Lambda u : K.O \mid O\{A\}$

Difference between standard CPS form and ML-CPS form

Standard CPS values	$W ::= x \mid \lambda x : A.N \mid \Lambda u : K.N$
Standard CPS terms	$N ::= W \mid NW \mid N\{A\}$
ML-CPS values	$X ::= x \mid \lambda x : A.O \mid \Lambda u : K.X$
ML-CPS terms	$O ::= X \mid OX \mid \Lambda u : K.O \mid O\{A\}$

ML-CPS form properties (Theorem 4.3)

The ML-CPS form has the following properties:

1. ML-CPS form is closed under CBV, CBN, ML-CBV, and ML-CBN reductions.
2. If O_1 is a ML-CPS term then $O_1 \hookrightarrow_{ml-cbv} O_2$ iff $O_1 \hookrightarrow_{ml-cbn} O_2$.
3. If O_1 is a ML-CPS term then $O_1 \hookrightarrow_{cbv} O_2$ iff $O_1 \hookrightarrow_{cbn} O_2$.
4. CBV, CBN, ML-CBV, or ML-CBN evaluation of well-typed, closed ML-CPS terms terminates in a ML-CPS value.

Proof

Looking at the definitions of ML-CBV and ML-CBN gives us the following evaluation strategy:

$$\begin{aligned} V &::= \quad X \\ R &::= \quad (\lambda x : A.O)V \mid (\Lambda u : K.V)\{A\} \\ E &::= \quad [\,] \mid EV \mid \Lambda u : K.E \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.O)V] &\quad\hookrightarrow\quad E[[V/x]O] \\ E[(\Lambda u : K.V)\{A\}] &\quad\hookrightarrow\quad E[[A/u]V] \end{aligned}$$

Proof

Looking at the definitions of ML-CBV and ML-CBN gives us the following evaluation strategy:

$$\begin{aligned} V &::= \; x \mid \lambda x : A.O \mid \Lambda u : K.X \\ R &::= \; (\lambda x : A.O)V \mid (\Lambda u : K.V)\{A\} \\ E &::= \; [\,] \mid EV \mid \Lambda u : K.E \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.O)V] &\hookrightarrow E[[V/x]O] \\ E[(\Lambda u : K.V)\{A\}] &\hookrightarrow E[[A/u]V] \end{aligned}$$

Proof

Compare the reductions of ML-CBV and ML-CBN:

$$\begin{array}{rcl} E[(\lambda x : A.M)V] & \hookrightarrow_{ml-cbv} & E[[V/x]M] \\ E[(\Lambda u : K.M)\{A\}] & \hookrightarrow_{ml-cbv} & E[[A/u]M] \end{array}$$

$$\begin{array}{rcl} E[(\lambda x : A.M_1)M_2] & \hookrightarrow_{ml-cbn} & E[[M_2/x]M_1] \\ E[(\Lambda u : K.M)\{A\}] & \hookrightarrow_{ml-cbn} & E[[A/u]M] \end{array}$$

Proof

Again, the Λ coincide.

And again, the argument for function application is a ML-CPS value.

So ML-CBV and ML-CBN also coincide for λ.

Strict CPS form

Why strict CPS form?

CBV/ML-CBV and CBN/ML-CBN do not coincide on terms in ML-CPS form.

Example:

$$\Lambda u : K.(\lambda x : A.x)c$$

This term is irreducible under CBV/CBN, but is reducible under ML-CBV/ML-CBN.

Strict CPS form definition

Strict CPS values	$Y ::= x \mid \lambda x : A.Q \mid \Lambda u : K.Y$
Strict CPS terms	$Q ::= Y \mid QY \mid Q\{A\}$

Differences

Standard CPS values	$W ::= x \mid \lambda x : A.N \mid \Lambda u : K.N$
Standard CPS terms	$N ::= W \mid NW \mid N\{A\}$
ML-CPS values	$X ::= x \mid \lambda x : A.O \mid \Lambda u : K.X$
ML-CPS terms	$O ::= X \mid OX \mid \Lambda u : K.O \mid O\{A\}$
Strict CPS values	$Y ::= x \mid \lambda x : A.Q \mid \Lambda u : K.Y$
Strict CPS terms	$Q ::= Y \mid QY \mid Q\{A\}$

Strict CPS form properties (Theorem 4.4)

1. Strict CPS form is closed under CBV, CBN, ML-CBV, and ML-CBN reductions.
2. If N_1 is a strict CPS term then if $N_1 \hookrightarrow N_2$ under one of CBV, CBN, ML-CBV, or ML-CBN, then $N_1 \hookrightarrow N_2$ under all of them.
3. CBV, CBN, ML-CBV, or ML-CBN evaluation of well-typed, closed strict CPS terms terminates in a strict CPS value.

Proof

Similar to the previous cases. This strategy is equivalent to all four (CBV, CBN, ML-CBV, ML-CBN) strategies on strict CPS terms:

$$\begin{aligned} V &::= \quad Y \\ R &::= \quad (\lambda x : A.Q)V \mid (\Lambda u : K.V)\{A\} \\ E &::= \quad [\,] \mid EV \mid E\{A\} \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.Q)V] &\quad \hookrightarrow \quad E[[V/x]Q] \\ E[(\Lambda u : K.V)\{A\}] &\quad \hookrightarrow \quad E[[A/u]V] \end{aligned}$$

Proof

Compare reductions for all 4 strategies:

$$\begin{aligned} E[(\lambda x : A.M)V] &\hookrightarrow_{cbv} E[[V/x]M] \\ E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{cbv} E[[A/u]M] \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M_1)M_2] &\hookrightarrow_{cbn} E[[M_2/x]M_1] \\ E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{cbn} E[[A/u]M] \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M)V] &\hookrightarrow_{ml-cbv} E[[V/x]M] \\ E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{ml-cbv} E[[A/u]M] \end{aligned}$$

$$\begin{aligned} E[(\lambda x : A.M_1)M_2] &\hookrightarrow_{ml-cbn} E[[M_2/x]M_1] \\ E[(\Lambda u : K.M)\{A\}] &\hookrightarrow_{ml-cbn} E[[A/u]M] \end{aligned}$$

Proof

Again, all strategies coincide for Λ.

Looking at the strict CPS terms ($Q ::= Y \mid QY \mid Q\{A\}$) we see that the argument of a function application is a strict CPS value.

So all strategies also coincide for λ.