Type Theory and Coq

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Principal Types and Type Checking

Overview of todays lecture

- Simple Type Theory à la Curry
  (versus Simple Type Theory à la Church)
- Principal Types algorithm
- Type checking dependent tytpe theory: λP

Recap: Simple type theory a la Church.

Formulation with contexts to declare the free variables:

$$x_1 : \sigma_1, x_2 : \sigma_2, \ldots, x_n : \sigma_n$$

is a context, usually denoted by Γ.

Derivation rules of $\lambda\!\to$ (à la Church):

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \sigma{\to}\tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \qquad \frac{\Gamma, x{:}\sigma \vdash P : \tau}{\Gamma \vdash \lambda x{:}\sigma.P : \sigma{\to}\tau}$$

$\Gamma \vdash_{\lambda\to} M : \sigma$ if there is a derivation using these rules with conclusion $\Gamma \vdash M : \sigma$

Recap: Formulas-as-Types (Curry, Howard)

There are two readings of a judgement $M : \sigma$

1. term as algorithm/program, type as specification:
   M is a function of type σ

2. type as a proposition, term as its proof:
   M is a proof of the proposition σ

- There is a one-to-one correspondence:

  typable terms in $\lambda{\rightarrow}$ $\simeq$ derivations in minimal proposition logic

- $x_1 : \tau_1, x_2 : \tau_2, \ldots, x_n : \tau_n \vdash M : \sigma$ can be read as
  M is a proof of σ from the assumptions $\tau_1, \tau_2, \ldots, \tau_n$.

Untyped λ-calculus

$$\Lambda ::= \mathsf{Var} \mid (\Lambda\,\Lambda) \mid (\lambda\mathsf{Var}.\Lambda)$$

Examples:

- $\mathbf{K} := \lambda x\, y.x$
- $\mathbf{S} := \lambda x\, y\, z.x\, z(y\, z)$
- $\omega := \lambda x.x\, x$
- $\Omega := \omega\, \omega$

$$\Omega \longrightarrow_\beta \Omega$$

Untyped λ-calculus is Turing complete
It's power lies in the fact that you can solve recursive equations:

Is there a term M such that

$$M\,x =_\beta x\,M\,x?$$

Is there a term M such that

$$M\,x =_\beta \mathbf{if}\,(\mathsf{Zero}\,x)\,\mathbf{then}\,1\,\mathbf{else}\,\mathsf{Mult}\,x\,(M\,(\mathsf{Pred}\,x))?$$

Yes, because we have a fixed point combinator:
- $\mathbf{Y} := \lambda f.(\lambda x.f\,(x\,x))(\lambda x.f\,(x\,x))$
Property:

$$Y\,f =_\beta f(Y\,f)$$

Why do we want to add types to λ-calculus?

- Types give a (partial) specification
- Typed terms can't go wrong (Milner) Subject Reduction property
- Typed terms always terminate
- The type checking algorithm detects (simple) mistakes

But: The compiler should compute the type information for us!
(Why would the programmer have to type all that?)

This is called a type assignment system, or also typing à la Curry:

For M an untyped term, the type system assigns a type σ to M (or not)

STT à la Church and à la Curry

$\lambda\!\rightarrow$ (à la Church):

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \sigma{\rightarrow}\tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \qquad \frac{\Gamma, x{:}\sigma \vdash P : \tau}{\Gamma \vdash \lambda x{:}\sigma.P : \sigma{\rightarrow}\tau}$$

$\lambda\!\rightarrow$ (à la Curry):

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \sigma{\rightarrow}\tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \qquad \frac{\Gamma, x{:}\sigma \vdash P : \tau}{\Gamma \vdash \lambda x.P : \sigma{\rightarrow}\tau}$$

Examples

- Typed Terms:

$$\lambda x : \alpha.\lambda y : (\beta \to \alpha) \to \alpha.y(\lambda z : \beta.x)$$

  has only the type $\alpha \to ((\beta \to \alpha) \to \alpha) \to \alpha$

- Type Assignment:

$$\lambda x.\lambda y.y(\lambda z.x)$$

  can be assigned the types

  - $\alpha \to ((\beta \to \alpha) \to \alpha) \to \alpha$
  - $(\alpha \to \alpha) \to ((\beta \to \alpha \to \alpha) \to \gamma) \to \gamma$
  - ...

  with $\alpha \to ((\beta \to \alpha) \to \gamma) \to \gamma$ being the principal type

Connection between Church and Curry typed STT

Definition The erasure map $|-|$ from STT à la Church to STT à la Curry is defined by erasing all type information.

$$\begin{array}{rcl} |x| & := & x \\ |M\,N| & := & |M|\,|N| \\ |\lambda x : \sigma.M| & := & \lambda x.|M| \end{array}$$

So, e.g.

$$|\lambda x : \alpha.\lambda y : (\beta{\to}\alpha){\to}\alpha.y(\lambda z : \beta.x)| = \lambda x.\lambda y.y(\lambda z.x)$$

Theorem If $M : \sigma$ in STT à la Church, then $|M| : \sigma$ in STT à la Curry.

Theorem If $P : \sigma$ in STT à la Curry, then there is an M such that $|M| \equiv P$ and $M : \sigma$ in STT à la Church.

Connection between Church and Curry typed STT

Definition The erasure map $|-|$ from STT à la Church to STT à la Curry is defined by erasing all type information.

$$
\begin{aligned}
|x| &:= x \\
|M\,N| &:= |M|\,|N| \\
|\lambda x:\sigma.M| &:= \lambda x.|M|
\end{aligned}
$$

Theorem If $P:\sigma$ in STT à la Curry, then there is an M such that $|M| \equiv P$ and $M:\sigma$ in STT à la Church.

Proof: by induction on derivations.

$$
\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \sigma{\rightarrow}\tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M N : \tau} \qquad \frac{\Gamma, x{:}\sigma \vdash P : \tau}{\Gamma \vdash \lambda x{:}\sigma.P : \sigma{\rightarrow}\tau}
$$

Example of computing a principal type

$$\lambda x^{\alpha}.\lambda y^{\beta}.\underbrace{y^{\beta}(\lambda z^{\gamma}.\overbrace{y^{\beta}x^{\alpha}}^{\delta})}_{\varepsilon}$$

1. Assign type vars to all variables: $x : \alpha, y : \beta, z : \gamma$.
2. Assign type vars to all applicative subterms: $y\,x : \delta$, $y(\lambda z.y\,x) : \varepsilon$.
3. Generate equations between types, necessary for the term to be typable: $\beta = \alpha{\to}\delta$ $\qquad$ $\beta = (\gamma{\to}\delta){\to}\varepsilon$
4. Find a most general unifier (a substitution) for the type vars that solves the equations: $\alpha := \gamma{\to}\varepsilon,\ \ \beta := (\gamma{\to}\varepsilon){\to}\varepsilon,\ \ \delta := \varepsilon$
5. The principal type of $\lambda x.\lambda y.y(\lambda z.yx)$ is now

$$(\gamma{\to}\varepsilon){\to}((\gamma{\to}\varepsilon){\to}\varepsilon){\to}\varepsilon$$

Exercise

Compute principal types for

- $\mathbf{S} := \lambda x.\lambda y.\lambda z.x\, z(y\, z)$
- $M := \lambda x.\lambda y.x(y(\lambda z.x\, z\, z))(y(\lambda z.x\, z\, z))$.

Principal Types: Preliminary Definitions

- A type substitution (or just substitution) is a map S from type variables to types. (Note: we can compose substitutions.)
- A unifier of the types σ and τ is a substitution that "makes σ and τ equal", i.e. an S such that $S(\sigma) = S(\tau)$
- A most general unifier (or mgu) of the types σ and τ is the "simplest substitution" that makes σ and τ equal, i.e. an S such that
  - $S(\sigma) = S(\tau)$
  - for all substitutions T such that $T(\sigma) = T(\tau)$ there is a substitution R such that $T = R \circ S$.

All these notions generalize to lists of types $\sigma_1, \ldots, \sigma_n$ in stead of pairs σ, τ.

Computing a most general unifier

There is an algorithm U that, when given types $\sigma_1, \ldots, \sigma_n$ outputs

- A most general unifier of $\sigma_1, \ldots, \sigma_n$, if $\sigma_1, \ldots, \sigma_n$ can be unified.
- "Fail" if $\sigma_1, \ldots, \sigma_n$ can't be unified.

- $U(\langle \alpha = \alpha, \ldots, \sigma_n = \tau_n \rangle) := U(\langle \sigma_2 = \tau_2, \ldots, \sigma_n = \tau_n \rangle)$.
- $U(\langle \alpha = \tau_1, \ldots, \sigma_n = \tau_n \rangle) :=$ "reject" if $\alpha \in \mathsf{FV}(\tau_1)$, $\tau_1 \neq \alpha$.
- $U(\langle \sigma_1 = \alpha, \ldots, \sigma_n = \tau_n \rangle) := U(\langle \alpha = \sigma_1, \ldots, \sigma_n = \tau_n \rangle)$
- $U(\langle \alpha = \tau_1, \ldots, \sigma_n = \tau_n \rangle) := [\alpha := V(\tau_1), V]$, if $\alpha \notin \mathsf{FV}(\tau_1)$,
  where V abbreviates
  $U(\langle \sigma_2[\alpha := \tau_1] = \tau_2[\alpha := \tau_1], \ldots, \sigma_n[\alpha := \tau_1] = \tau_n[\alpha := \tau_1] \rangle)$.
- $U(\langle \mu \to \nu = \rho \to \xi, \ldots, \sigma_n = \tau_n \rangle) := U(\langle \mu = \rho, \nu = \xi, \ldots, \sigma_n = \tau_n \rangle)$

Principal type: Definition

Definition σ is a principal type for the closed untyped λ-term M if

- $M : \sigma$ in STT à la Curry
- for all types τ, if $M : \tau$, then $\tau = S(\sigma)$ for some substitution S.

  A principal type is unique up to renaming of type variables.

  Both $\alpha \to \alpha$ and $\beta \to \beta$ are principal type of $\lambda x.x$.

Principal Types Theorem

Theorem There is an algorithm PT that, when given a closed untyped λ-term M, outputs

A principal type σ of M	if M is typable in STT à la Curry,
"Fail"	if M is not typable in STT à la Curry.

This can be extended to open untyped λ-terms: There is an algorithm PP that, when given an untyped λ-term M, outputs

A principal pair (Γ, σ) of M	if M is typable in STT à la Curry,
"Fail"	if M is not typable in STT à la Curry.

Definition (Γ, σ) is a principal pair for M if $\Gamma \vdash M : \sigma$ and for every typing $\Delta \vdash M : \tau$ there is a substitution S such that $\tau = S(\sigma)$ and $\Delta = S(\Gamma)$.

Typical problems one would like to have an algorithm for

$M : \sigma?$	Type Checking Problem	TCP
$M : ?$	Type Synthesis Problem	TSP
$? : \sigma$	Type Inhabitation Problem (by a closed term)	TIP

For $\lambda\!\rightarrow$, all these problems are decidable,
both for the Curry style and for the Church style presentation.

- TCP and TSP are (usually) equivalent: To solve $MN : \sigma$, one has to solve $N :?$ (and if this gives answer τ, solve $M : \tau\rightarrow\sigma$).
- For Curry systems, TCP and TSP soon become undecidable beyond $\lambda\!\rightarrow$.
- TIP is undecidable for most extensions of $\lambda\!\rightarrow$, as it corresponds to provability in some logic.

Rules for λP: axiom, application, abstraction, product

$$\frac{}{\vdash * : \Box}$$

$$\frac{\Gamma \vdash M : \Pi x : A.\, B \qquad \Gamma \vdash N : A}{\Gamma \vdash M N : B[x := N]}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}$$

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

Rules for λP: weakening, variable, conversion

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}$$

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \qquad \text{with } B =_\beta B'$$

Properties of λP

- Uniqueness of types
  If $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M : \tau$, then $\sigma =_\beta \tau$.

- Subject Reduction
  If $\Gamma \vdash M : \sigma$ and $M \longrightarrow_\beta N$, then $\Gamma \vdash N : \sigma$.

- Strong Normalization
  If $\Gamma \vdash M : \sigma$, then all β-reductions from M terminate.

Proof of SN is by defining a reduction preserving map from λP to $\lambda\!\rightarrow$.

Decidability Questions

$$\Gamma \vdash M : \sigma? \quad \text{TCP}$$

$$\Gamma \vdash M : ? \quad \text{TSP}$$

$$\Gamma \vdash ? : \sigma \quad \text{TIP}$$

For λP:

- TIP is undecidable
  (Equivalent to provability in minimal predicate logic.)
- TCP/TSP: simultaneously with Context checking

Type Checking algorithm for λP

Define algorithms $\mathrm{Ok}(-)$ and $\mathrm{Type}_{\_}(-)$ simultaneously:

- $\mathrm{Ok}(-)$ takes a context and returns 'true' or 'false'
- $\mathrm{Type}_{\_}(-)$ takes a context and a term and returns a term or 'false'.

Definition. The type synthesis algorithm $\mathrm{Type}_{\_}(-)$ is sound if

$$\mathrm{Type}_{\Gamma}(M) = A \quad \Rightarrow \quad \Gamma \vdash M : A$$

for all Γ and M.

Definition. The type synthesis algorithm $\mathrm{Type}_{\_}(-)$ is complete if

$$\Gamma \vdash M : A \quad \Rightarrow \quad \mathrm{Type}_{\Gamma}(M) =_{\beta} A$$

for all Γ, M and A.

$$
\begin{array}{rcl}
\mathrm{Ok}(<>) & = & \text{'true'} \\[2em]
\mathrm{Ok}(\Gamma, x{:}A) & = & \mathrm{Type}_\Gamma(A) \in \{*, \mathbf{kind}\}, \\[2em]
\mathrm{Type}_\Gamma(x) & = & \text{if } \mathrm{Ok}(\Gamma) \text{ and } x{:}A \in \Gamma \text{ then } A \text{ else 'false'}, \\[2em]
\mathrm{Type}_\Gamma(\mathbf{type}) & = & \text{if } \mathrm{Ok}(\Gamma) \text{ then } \mathbf{kind} \text{ else 'false'}, \\[2em]
\mathrm{Type}_\Gamma(MN) & = & \text{if } \mathrm{Type}_\Gamma(M) = C \text{ and } \mathrm{Type}_\Gamma(N) = D \\
 & & \quad \text{then} \quad \text{if } C \twoheadrightarrow_\beta \Pi x{:}A.B \text{ and } A =_\beta D \\
 & & \qquad\qquad\quad \text{then } B[x := N] \text{ else 'false'} \\
 & & \quad \text{else} \quad\ \ \text{'false'},
\end{array}
$$

$$
\begin{aligned}
\mathrm{Type}_{\Gamma}(\lambda x{:}A.M) \quad &= \quad \text{if } \mathrm{Type}_{\Gamma,x:A}(M) = B \\
&\quad\ \text{then} \qquad\qquad \text{if } \mathrm{Type}_{\Gamma}(\Pi x{:}A.B) \in \{\mathbf{type}, \mathbf{kind}\} \\
&\qquad\qquad\qquad\qquad\ \text{then } \Pi x{:}A.B \text{ else `false'} \\
&\quad\ \text{else `false'}, \\
\mathrm{Type}_{\Gamma}(\Pi x{:}A.B) \quad &= \quad \text{if } \mathrm{Type}_{\Gamma}(A) = \mathbf{type} \text{ and } \mathrm{Type}_{\Gamma,x:A}(B) = s \\
&\quad\ \text{then } s \text{ else `false'}
\end{aligned}
$$

Soundness and Completeness

Soundness

$$\mathrm{Type}_\Gamma(M) = A \;\Rightarrow\; \Gamma \vdash M : A$$

Completeness

$$\Gamma \vdash M : A \;\Rightarrow\; \mathrm{Type}_\Gamma(M) =_\beta A$$

As a consequence:

$$\mathrm{Type}_\Gamma(M) = \text{‘false’} \;\Rightarrow\; M \text{ is not typable in } \Gamma$$

NB 1. Completeness only makes sense if types are uniqueness upto $=_\beta$ (Otherwise: let $\mathrm{Type}_{\_}(-)$ generate a set of possible types)

NB 2. Completeness only implies that Type terminates on all well-typed terms. We want that Type terminates on all pseudo terms.

Termination

We want $\mathrm{Type}_{\_}(-)$ to terminate on all inputs.

Interesting cases: λ-abstraction and application:

$$\mathrm{Type}_{\Gamma}(\lambda x{:}A.M) \quad = \quad \text{if } \mathrm{Type}_{\Gamma,x:A}(M) = B$$

$$\text{then} \quad \text{if } \mathrm{Type}_{\Gamma}(\Pi x{:}A.B) \in \{\mathbf{type}, \mathbf{kind}\}$$

$$\text{then } \Pi x{:}A.B \text{ else 'false'}$$

$$\text{else 'false'},$$

! Recursive call is not on a smaller term!

Replace the side condition

$$\text{if } \mathrm{Type}_{\Gamma}(\Pi x{:}A.B) \in \{\mathbf{type}, \mathbf{kind}\}$$

by

$$\text{if } \mathrm{Type}_{\Gamma}(A) \in \{\mathbf{type}\}$$

Termination

We want $\mathrm{Type}_{\_}(-)$ to terminate on all inputs.

Interesting cases: λ-abstraction and application:

$$
\begin{array}{rcl}
\mathrm{Type}_\Gamma(MN) & = & \text{if } \mathrm{Type}_\Gamma(M) = C \text{ and } \mathrm{Type}_\Gamma(N) = D \\
& & \text{then} \quad \text{if } C \twoheadrightarrow_\beta \Pi x{:}A.B \text{ and } A =_\beta D \\
& & \qquad\quad \text{then } B[x := N] \text{ else 'false'} \\
& & \text{else} \quad \text{'false'},
\end{array}
$$

! Need to decide β-reduction and β-equality!

For this case, termination follows from soundness of Type and the decidability of equality on well-typed terms (using SN and CR).