For substitution (substitute N for x in M) one sometimes writes $M[x := N]$ (e.g. Takahashi) and sometimes $M[N/x]$ (the exercise sheet).

Exercise Church-Rosser (1)

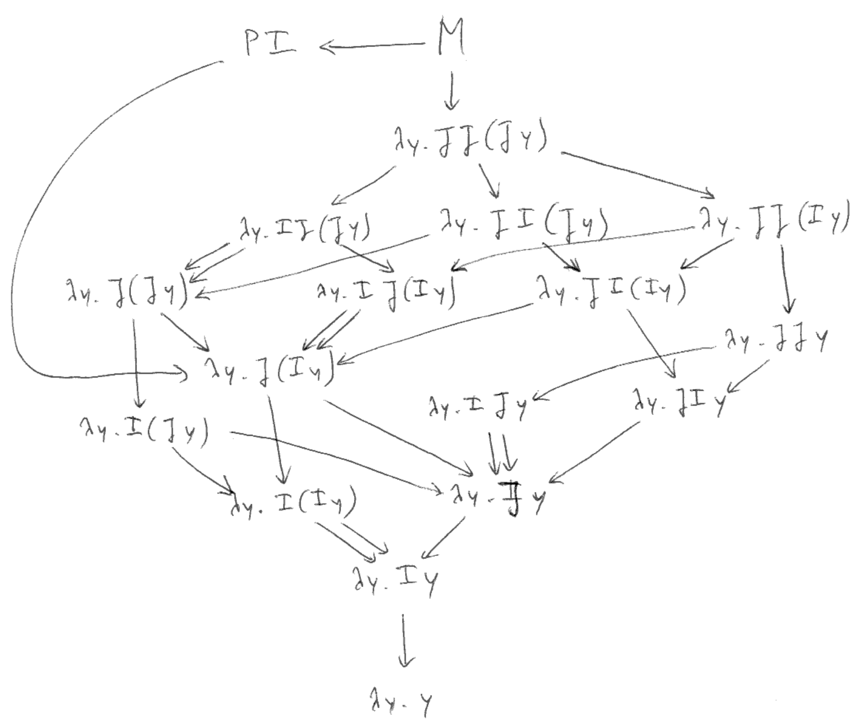
(1) $M = (\lambda xy.\, xx(xy))(II)$

$P = \lambda xy.\, xx(xy)$

$J = II$

(a) We omit duplicates in the graph.

Note that $\lambda y.\, II(xI)$ is just $\lambda y.\, J(xI)$ etc.



(b) $M^* = \lambda y.\, II(Iy)$

$(M^*)^* = \lambda y.\, Iy$

1 c

2

$$\dfrac{\dfrac{\dfrac{\dfrac{x \Rightarrow x \quad x \Rightarrow x}{xx \Rightarrow xx} \qquad \dfrac{x \Rightarrow x \quad y \Rightarrow y}{xy \Rightarrow xy}}{xx(xy) \Rightarrow xx(xy)}}{\lambda y. xx(xy) \Rightarrow \lambda y. xx(xy)} \qquad \dfrac{z \Rightarrow z \qquad \lambda z.z \Rightarrow \lambda z.z}{(\lambda z.z) I \Rightarrow I}}{(\lambda xy. xx(xy))(II) \Rightarrow \lambda y. II(Iy)}$$

$M \Rightarrow M^*$

$\boxed{I = \lambda z.z}$

$$\dfrac{\dfrac{\dfrac{z \Rightarrow z \qquad \dfrac{z \Rightarrow z}{I \Rightarrow I}}{II \Rightarrow I} \qquad \dfrac{z \Rightarrow z \qquad y \Rightarrow y}{Iy \Rightarrow y}}{II(Iy) \Rightarrow Iy}}{\lambda y. II(Iy) \Rightarrow \lambda y. Iy}$$

$\boxed{I = \lambda z.z}$

$\boxed{M^* \Rightarrow (M^*)^*}$

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2 a Change the cases $(\beta 3^*)$ and $(\beta 4^*)$ to the following

$(\beta 5^*)$ If $M_1^* \equiv \lambda x. P$, then $(M_1 M_2)^* \equiv P[x := M_2^*]$

b

$$\dfrac{M \Rightarrow \lambda x. P \qquad N \Rightarrow N'}{MN \Rightarrow \cancel{P[N'/x]}\ P[x := N']}$$

IH $\lambda x. P \Rightarrow M^*$, $N' \Rightarrow N^*$

To prove: $\cancel{P[N'/x] \Rightarrow (MN)^*}$ $P[x := N'] \Rightarrow (MN)^*$

From $\lambda x. P \Rightarrow M^*$ we deduce that $M^* = \lambda x. Q$ with $P \Rightarrow Q$

So $(MN)^* = Q[x := N^*]$.

we have $P \Rightarrow Q$ and $N' \Rightarrow N^*$, so by substitution (property (3) in the Takahashi paper) we conclude $P[x := N'] \Rightarrow Q[x := N^*]$

$\qquad \| $

$\qquad \quad (MN)^*$ ⊠

③ (a) ③

$$\dfrac{\dfrac{y \Rightarrow y}{\lambda x. y\, x \Rightarrow y} \qquad I \Rightarrow I}{(\lambda y. \lambda x. y\, x)\, I \Rightarrow I}$$

$$I \;\|\; y[y := I]$$

We don't have $(\lambda y. \lambda x. y\, x)\, I \Rightarrow I$

because if this were derivable, a derivation has to have the following shape

$$\dfrac{\dfrac{(*)}{\lambda x. y\, x \Rightarrow M'} \qquad I \Rightarrow I}{(\lambda y. \lambda x\, y\, x)\, I \Longrightarrow M'[y := I]}$$

$$M'[y := I] \;\|\; I$$

Note $M'[y = I] \equiv I$ can be because of

(i) $M' \equiv y$

(ii) $M' \equiv I$

In case (i) we must have a derivation $\dfrac{(*)}{\lambda x. y\, x \Rightarrow y}$

but that can't be, because a λ-abstraction only parallel reduces to another λ-abstraction

In case (ii) we must have $\dfrac{(*)}{\lambda x\, y\, x \Rightarrow \lambda x. x}$

and so $y\, x \Rightarrow x$, which is not the case either.

So: we can't have such a derivation ⊠

(b) $(\lambda x. M)^* = \begin{cases} P^* & \text{if } M = P x \text{ with } x \notin FV(P) \\ \lambda x. M^* & \text{otherwise.} \end{cases}$