Course: Type Theory and Coq

Exercises on Normalization

See the Slides and the Course Notes by Herman Geuvers for the definitions.

1. In the proof of WN for $\lambda\!\to$, the height of a type $h(\sigma)$ is defined by

   - $h(\alpha) := 0$
   - $h(\sigma_1 \to \ldots \to \sigma_n \to \alpha) := \max(h(\sigma_1), \ldots, h(\sigma_n)) + 1$.

   Prove that this is the same as taking as the second clause

   - $h(\sigma \to \tau) := \max(h(\sigma) + 1, h(\tau))$.

2. In the proof of WN for $\lambda\!\to$, it is stated that, if $M \longrightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that is not contained in another redex of maximum height, then this does not create a new redex of maximum height.

   Show that this holds for the case

   $$\begin{array}{rcl} M & = & (\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I}))(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I})) \\ & \longrightarrow_\beta & (\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))(\lambda v : B.(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))\,\mathbf{I}) \quad = \quad N \end{array}$$

   where $B = \alpha \to \alpha$, $C = B \to B$ and $A = C \to B$.

   Also show that $m(M) >_l m(N)$.

3. Prove that *type reduction* is SN for $\lambda 2$ a la Church. (Define a simple measure on terms that decreases with type reduction.)

4. Prove for that for $A, B \in \mathrm{SAT}$, $A \to B \in \mathrm{SAT}$.
   (Here, $A \to B := \{M | \forall N \in A (M\,N \in B)\}$. Check the slides or course notes for the definition of SAT, the collection of *saturated sets*.)

5. The Soundness of the saturated sets model for $\lambda 2$ is proved by induction on the derivation. Do the case for the $\forall$-introduction rule.