

## Exercises on Principal Types

①

**1a**  $(\alpha \rightarrow \beta) \rightarrow \gamma = \alpha \rightarrow \beta \rightarrow \gamma$   
 $\Leftrightarrow \begin{cases} \alpha \rightarrow \beta = \alpha \\ \gamma = \beta \rightarrow \gamma \end{cases} \Leftrightarrow \text{FAIL} \quad \underline{\text{no mgu}}$

**1b**  $(\alpha \rightarrow \beta) \rightarrow \gamma = \text{UNATYPABLE } \gamma \rightarrow \alpha \rightarrow \beta$   
 $\Leftrightarrow \begin{cases} \alpha \rightarrow \beta = \gamma \\ \gamma = \alpha \rightarrow \beta \end{cases} \Leftrightarrow \begin{cases} \gamma = \alpha \rightarrow \beta \end{cases} \quad \underline{\text{so mgu is } \gamma := \alpha \rightarrow \beta}$

NOTE  $\alpha \rightarrow \beta \rightarrow \gamma$  means  $\alpha \rightarrow (\beta \rightarrow \gamma)$

We use the method as explained in the course-video.

One can also use the mathematical definition of  $U$ , as in the course notes:  $U\langle (\alpha \rightarrow \beta) \rightarrow \gamma, \gamma \rightarrow \alpha \rightarrow \beta \rangle = [\gamma := \alpha \rightarrow \beta]$ .

**2a** Answer: pt of  $S$  is  $(\gamma \rightarrow \beta \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta) \rightarrow \gamma \rightarrow \alpha$

**3a**  $\lambda z. \lambda x. \lambda y. z (x (\lambda y. y x)) \quad : \alpha \rightarrow \beta \rightarrow \eta$

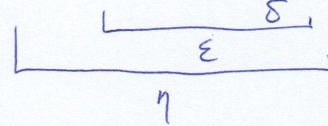
$\begin{array}{c} \delta \\ \hline \varepsilon \\ \hline \eta \end{array}$

Equations:

$$\begin{cases} \gamma = \beta \rightarrow \delta \\ \beta = (\gamma \rightarrow \delta) \rightarrow \varepsilon \\ \alpha = \varepsilon \rightarrow \eta \end{cases} \Leftrightarrow \begin{cases} \gamma = \beta \rightarrow \delta \\ \beta = (\beta \rightarrow \delta) \rightarrow \delta \rightarrow \varepsilon \\ \alpha = \varepsilon \rightarrow \eta \end{cases} \Leftrightarrow \text{FAIL} \quad \underline{\text{not typable}}$$

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$$\lambda z^x \lambda x^y . z (x (\lambda y^r . y z)) : \alpha \rightarrow \beta \rightarrow \eta \quad (2)$$



Equations:

$$\begin{cases} \beta = \alpha \rightarrow \delta \\ \beta = (\beta \rightarrow \delta) \rightarrow \epsilon \\ \alpha = \epsilon \rightarrow \eta \end{cases} \Leftrightarrow \begin{cases} \beta = \alpha \rightarrow \delta \\ \beta = ((\alpha \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon \\ \alpha = \epsilon \rightarrow \eta \end{cases}$$

$$\Leftrightarrow \begin{cases} \beta = \alpha \rightarrow \delta \\ \beta = (((\epsilon \rightarrow \eta) \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon \\ \alpha = \epsilon \rightarrow \eta \end{cases}$$

So in  $\eta$  is the substitution  $\left[ \beta := (\epsilon \rightarrow \eta) \rightarrow \delta, \beta := (((\epsilon \rightarrow \eta) \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon, \alpha := \epsilon \rightarrow \eta \right]$

$$\text{So PT is } (\epsilon \rightarrow \eta) \rightarrow (((\epsilon \rightarrow \eta) \rightarrow \delta) \rightarrow \delta) \rightarrow \epsilon \rightarrow \eta$$

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4 Answer: PT is  $(\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \beta$



5a

$$\lambda x (\lambda y. x(x y)) (\lambda u. \lambda v. u) : \alpha \rightarrow \mu \quad (3)$$

$\underbrace{\hspace{10em}}_{\eta}$   
 $\underbrace{\hspace{5em}}_{\epsilon}$   
 $\mu$

Equations:

$$\begin{cases} \alpha = \beta \rightarrow \epsilon \\ \alpha = \epsilon \rightarrow \eta \\ \beta \rightarrow \eta = (\delta \rightarrow \delta \rightarrow \delta) \rightarrow \mu \end{cases} \xrightarrow{2 \text{ steps}} \begin{cases} \alpha = \beta \rightarrow \epsilon \\ \beta \rightarrow \epsilon = \epsilon \rightarrow \eta \\ \beta = \delta \rightarrow \delta \rightarrow \delta \\ \eta = \mu \end{cases} \iff$$

$$\begin{cases} \alpha = \beta \rightarrow \epsilon \\ \mu = \epsilon \\ \epsilon = \eta \\ \beta = \delta \rightarrow \delta \rightarrow \delta \\ \eta = \mu \end{cases} \xrightarrow{3 \text{ steps}} \begin{cases} \alpha = (\delta \rightarrow \delta \rightarrow \delta) \rightarrow \mu \\ \beta = \delta \rightarrow \delta \rightarrow \delta \\ \epsilon = \mu \\ \eta = \mu \end{cases}$$

$$\iff \begin{cases} \alpha = (\delta \rightarrow \delta \rightarrow \delta) \rightarrow (\delta \rightarrow \delta \rightarrow \delta) \\ \mu = \delta \rightarrow \delta \rightarrow \delta \\ \epsilon = \delta \rightarrow \delta \rightarrow \delta \\ \beta = \delta \rightarrow \delta \rightarrow \delta \\ \eta = \delta \rightarrow \delta \rightarrow \delta \end{cases} \quad \text{This is the mgn.}$$

$$\underline{5a} \quad PT = ((\delta \rightarrow \delta \rightarrow \delta) \rightarrow (\delta \rightarrow \delta \rightarrow \delta)) \rightarrow (\delta \rightarrow \delta \rightarrow \delta)$$

5b is not typable.