Lineair substructural type system

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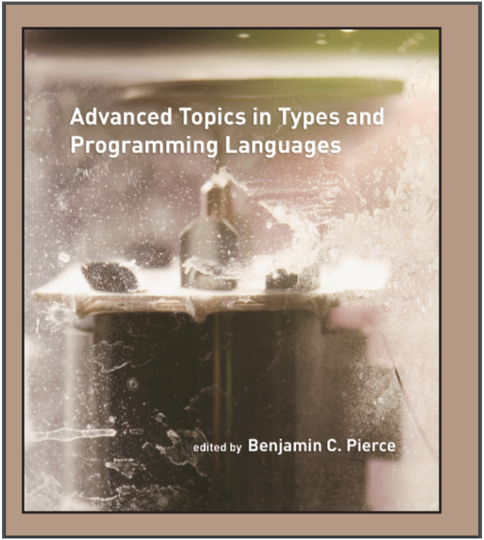
Outline

Introduction



1. Writer: David Walker
2. Article published in 2005

Introduction

Syntax

b	::=		*booleans:*
		true	*true*
		false	*false*
t	::=		*terms:*
		x	*variable*
		b	*boolean*
		if t then t else t	*conditional*
		λx:T.t	*abstraction*
		t t	*application*
T	::=		*types:*
		Bool	*booleans*
		T→T	*type of functions*
Γ	::=		*contexts:*
		$\varnothing$	*empty context*
		Γ, x:T	*term variable binding*

Typing $\boxed{\Gamma \vdash t : T}$

$$\frac{}{\Gamma_1, x{:}T, \Gamma_2 \vdash x : T} \quad \text{(T-VAR)}$$

$$\frac{}{\Gamma \vdash b : \mathtt{Bool}} \quad \text{(T-BOOL)}$$

$$\frac{\Gamma \vdash t_1 : \mathtt{Bool} \qquad \Gamma \vdash t_2 : T \qquad \Gamma \vdash t_3 : T}{\Gamma \vdash \mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t3 : T} \quad \text{(T-IF)}$$

$$\frac{\Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash \lambda x{:}T_1.t_2 : T_1 \to T_2} \quad \text{(T-ABS)}$$

$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \quad \text{(T-APP)}$$

Introduction

Lemma [Exchange]

$$\frac{\Gamma_1,\ x_1 : \mathrm{T}_1,\ x_2 : \mathrm{T}_2,\ \Gamma_2 \vdash t : \mathrm{T}}{\Gamma_1,\ x_2 : \mathrm{T}_2,\ x_1 : \mathrm{T}_1,\ \Gamma_2 \vdash t : \mathrm{T}}$$

Lemma [Weakening]

$$\frac{\Gamma_1,\ \Gamma_2 \vdash t : \mathrm{T}}{\Gamma_1,\ x : \mathrm{T},\ \Gamma_2 \vdash t : \mathrm{T}}$$

Lemma [Contraction]

$$\frac{\Gamma_1,\ x_2 : \mathrm{T}_1,\ x_3 : \mathrm{T}_1, \Gamma_2 \vdash t : \mathrm{T}_2}{\Gamma_1,\ x_1 : \mathrm{T}_1,\ \Gamma_2 \vdash [x_2 \mapsto x_1][x_3 \mapsto x_1]t : \mathrm{T}_2}$$

Introduction

Lemma [Exchange]

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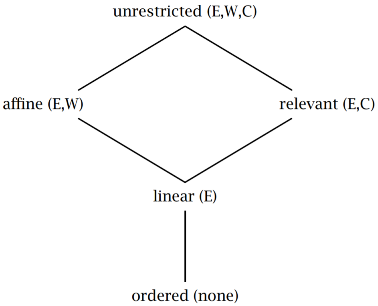

Introduction

Lemma [Exchange]

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$$\frac{\Gamma \vdash t_1 : \mathtt{Bool} \qquad \Gamma \vdash t_2 : T \qquad \Gamma \vdash t_3 : T}{\Gamma \vdash \mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t3 : T} \quad \text{(T-IF)}$$

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$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \qquad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}} \quad \text{(T-APP)}$$

Syntax

Syntax

`q`	::=		*qualifiers:*
		`lin`	*linear*
		`un`	*unrestricted*
`b`	::=		*booleans:*
		`true`	*true*
		`false`	*false*
`t`	::=		*terms:*
		`x`	*variable*
		`q b`	*boolean*
		`if t then t else t`	*conditional*
		`q <t,t>`	*pair*
		`split t as x,y in t`	*split*
		`q λx:T.t`	*abstraction*
		`t t`	*application*
`P`	::=		*pretypes:*
		`Bool`	*booleans*
		`T*T`	*pairs*
		`T→T`	*functions*
`T`	::=		*types:*
		`q P`	*qualified pretype*
Γ	::=		*contexts:*
		$\varnothing$	*empty context*
		Γ, `x:T`	*term variable binding*

Syntax

Further requirements:

1. A linear variable can only be used once.
2. An unresricted data structure cannot contain a linear data structure.
3. Examples of mistakes:
   - $(\lambda z.\lambda y. < \texttt{free}\ z,\ \texttt{free}\ y >)x\ x.$
   - $\texttt{let}\ z = un < x, 3 >\ \texttt{in}$
     $\texttt{split}\ z\ \texttt{as}\ x_1, \_\ \texttt{in}$
     $\texttt{split}\ z\ \texttt{as}\ x_2, \_\ \texttt{in}$
     $< \texttt{free}\ x_1, \texttt{free}\ x_2 >$

Syntax

Solution: context splitting

Context Split $\boxed{\Gamma = \Gamma_1 \circ \Gamma_2}$

$$\varnothing = \varnothing \circ \varnothing \quad \text{(M-EMPTY)}$$

$$\frac{\Gamma = \Gamma_1 \circ \Gamma_2}{\Gamma, \texttt{x:un P} = (\Gamma_1, \texttt{x:un P}) \circ (\Gamma_2, \texttt{x:un P})} \quad \text{(M-UN)}$$

$$\frac{\Gamma = \Gamma_1 \circ \Gamma_2}{\Gamma, \texttt{x:lin P} = (\Gamma_1, \texttt{x:lin P}) \circ \Gamma_2} \quad \text{(M-LIN1)}$$

$$\frac{\Gamma = \Gamma_1 \circ \Gamma_2}{\Gamma, \texttt{x:lin P} = \Gamma_1 \circ (\Gamma_2, \texttt{x:lin P})} \quad \text{(M-LIN2)}$$

Typing

Typing $\boxed{\Gamma \vdash t : T}$

$$\frac{un\ (\Gamma_1, \Gamma_2)}{\Gamma_1, x{:}T, \Gamma_2 \vdash x : T}\ \text{(T-VAR)}$$

$$\frac{un\ (\Gamma)}{\Gamma \vdash q\ b : q\ Bool}\ \text{(T-BOOL)}$$

$$\frac{\Gamma_1 \vdash t_1 : q\ Bool \quad \Gamma_2 \vdash t_2 : T \quad \Gamma_2 \vdash t_3 : T}{\Gamma_1 \circ \Gamma_2 \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t_3 : T}\ \text{(T-IF)}$$

$$\frac{\Gamma_1 \vdash t_1 : T_1 \quad \Gamma_2 \vdash t_2 : T_2 \quad q(T_1) \quad q(T_2)}{\Gamma_1 \circ \Gamma_2 \vdash q\ \langle t_1, t_2 \rangle : q\ (T_1 {*} T_2)}\ \text{(T-PAIR)}$$

$$\frac{\Gamma_1 \vdash t_1 : q\ (T_1 {*} T_2) \quad \Gamma_2, x{:}T_1, y{:}T_2 \vdash t_2 : T}{\Gamma_1 \circ \Gamma_2 \vdash \text{split } t_1 \text{ as } x,y \text{ in } t_2 : T}\ \text{(T-SPLIT)}$$

$$\frac{q(\Gamma) \quad \Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash q\ \lambda x{:}T_1.t_2 : q\ T_1 \to T_2}\ \text{(T-ABS)}$$

$$\frac{\Gamma_1 \vdash t_1 : q\ T_{11} \to T_{12} \quad \Gamma_2 \vdash t_2 : T_{11}}{\Gamma_1 \circ \Gamma_2 \vdash t_1\ t_2 : T_{12}}\ \text{(T-APP)}$$

Typing of lineair substructural type system.

Typing $\boxed{\Gamma \vdash t : T}$

$$\frac{}{\Gamma_1, x{:}T, \Gamma_2 \vdash x : T}\ \text{(T-VAR)}$$

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$$\frac{\Gamma \vdash t_1 : Bool \quad \Gamma \vdash t_2 : T \quad \Gamma \vdash t_3 : T}{\Gamma \vdash \text{if } t_1 \text{ then } t_2 \text{ else } t3 : T}\ \text{(T-IF)}$$

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$$\frac{\Gamma \vdash t_1 : T_{11} \to T_{12} \quad \Gamma \vdash t_2 : T_{11}}{\Gamma \vdash t_1\ t_2 : T_{12}}\ \text{(T-APP)}$$

Typing of Symply typed lambda calculus.

Typing

Typing $\boxed{\Gamma \vdash t : T}$

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$$\frac{\Gamma_1 \vdash t_1 : q\ \mathtt{Bool} \qquad \Gamma_2 \vdash t_2 : T \qquad \Gamma_2 \vdash t_3 : T}{\Gamma_1 \circ \Gamma_2 \vdash \mathtt{if}\ t_1\ \mathtt{then}\ t_2\ \mathtt{else}\ t_3 : T} \quad \text{(T-IF)}$$

$$\frac{\Gamma_1 \vdash t_1 : T_1 \qquad \Gamma_2 \vdash t_2 : T_2 \qquad q(T_1) \qquad q(T_2)}{\Gamma_1 \circ \Gamma_2 \vdash q\ \langle t_1, t_2 \rangle : q\ (T_1 {*} T_2)} \quad \text{(T-PAIR)}$$

$$\frac{\Gamma_1 \vdash t_1 : q\ (T_1 {*} T_2) \qquad \Gamma_2, x{:}T_1, y{:}T_2 \vdash t_2 : T}{\Gamma_1 \circ \Gamma_2 \vdash \mathtt{split}\ t_1\ \mathtt{as}\ x,y\ \mathtt{in}\ t_2 : T} \quad \text{(T-SPLIT)}$$

$$\frac{q(\Gamma) \qquad \Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash q\ \lambda x{:}T_1.t_2 : q\ T_1 \rightarrow T_2} \quad \text{(T-ABS)}$$

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Typing of lineair substructural type system.

Definition

1. $q(T)$ if and only if $T = q'\ P$ and $q \sqsubseteq q'$
2. $q(\Gamma)$ if and only if $(x{:}T) \in \Gamma$ implies $q(T)$

Typing

Typing $\boxed{\Gamma \vdash t : T}$

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$$\frac{\Gamma_1 \vdash t_1 : q\ Bool \quad \Gamma_2 \vdash t_2 : T \quad \Gamma_2 \vdash t_3 : T}{\Gamma_1 \circ \Gamma_2 \vdash if\ t_1\ then\ t_2\ else\ t_3 : T} \quad \text{(T-IF)}$$

$$\frac{\Gamma_1 \vdash t_1 : T_1 \quad \Gamma_2 \vdash t_2 : T_2 \quad q(T_1) \quad q(T_2)}{\Gamma_1 \circ \Gamma_2 \vdash q <t_1, t_2> : q\ (T_1 {*} T_2)} \quad \text{(T-PAIR)}$$

$$\frac{\Gamma_1 \vdash t_1 : q\ (T_1 {*} T_2) \quad \Gamma_2, x{:}T_1, y{:}T_2 \vdash t_2 : T}{\Gamma_1 \circ \Gamma_2 \vdash split\ t_1\ as\ x,y\ in\ t_2 : T} \quad \text{(T-SPLIT)}$$

$$\frac{q(\Gamma) \quad \Gamma, x{:}T_1 \vdash t_2 : T_2}{\Gamma \vdash q\ \lambda x{:}T_1.t_2 : q\ T_1 \rightarrow T_2} \quad \text{(T-ABS)}$$

$$\frac{\Gamma_1 \vdash t_1 : q\ T_{11} \rightarrow T_{12} \quad \Gamma_2 \vdash t_2 : T_{11}}{\Gamma_1 \circ \Gamma_2 \vdash t_1\ t_2 : T_{12}} \quad \text{(T-APP)}$$

Typing of lineair substructural type system.

Typing $\boxed{\Gamma \vdash t : T}$

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Typing of Symply typed lambda calculus.

Typing

Lemma [Exchange]

$$\frac{\Gamma_1,\ x_1 : \mathrm{T}_1,\ x_2 : \mathrm{T}_2,\ \Gamma_2 \vdash t : \mathrm{T}}{\Gamma_1,\ x_2 : \mathrm{T}_2,\ x_1 : \mathrm{T}_1,\ \Gamma_2 \vdash t : \mathrm{T}}$$

Lemma [Unrestricted Weakening]

$$\frac{\Gamma \vdash t : \mathrm{T}}{\Gamma,\ x_1 : \mathrm{un}\ P_1 \vdash t : \mathrm{T}}$$

Lemma [Unrestricted Contraction]

$$\frac{\Gamma,\ x_2 : \mathrm{un\ P}_1,\ x_3 : \mathrm{un\ P}_1 \vdash t : \mathrm{T}_3}{\Gamma,\ x_1 : \mathrm{un\ P}_1 \vdash [x_2 \mapsto x_1][x_3 \mapsto x_1]t : \mathrm{T}_3}$$

Algorithmic Type Checking

Algorithmic Typing $\boxed{\Gamma_{in} \vdash \mathtt{t} : \mathtt{T}; \Gamma_{out}}$

$$\Gamma_1, \mathtt{x{:}un\ P}, \Gamma_2 \vdash \mathtt{x} : \mathtt{un\ P}; \Gamma_1, \mathtt{x{:}un\ P}, \Gamma_2 \qquad \text{(A-UVar)}$$

$$\Gamma_1, \mathtt{x{:}lin\ P}, \Gamma_2 \vdash \mathtt{x} : \mathtt{lin\ P}; \Gamma_1, \Gamma_2 \qquad \text{(A-LVar)}$$

$$\Gamma \vdash \mathtt{q\ b} : \mathtt{q\ Bool}; \Gamma \qquad \text{(A-Bool)}$$

$$\frac{\Gamma_1 \vdash \mathtt{t_1} : \mathtt{q\ Bool}; \Gamma_2 \qquad \Gamma_2 \vdash \mathtt{t_2} : \mathtt{T}; \Gamma_3 \qquad \Gamma_2 \vdash \mathtt{t_3} : \mathtt{T}; \Gamma_3}{\Gamma_1 \vdash \mathtt{if\ t_1\ then\ t_2\ else\ t_3} : \mathtt{T}; \Gamma_3} \qquad \text{(A-If)}$$

$$\frac{\Gamma_1 \vdash \mathtt{t_1} : \mathtt{T_1}; \Gamma_2 \qquad \Gamma_2 \vdash \mathtt{t_2} : \mathtt{T_2}; \Gamma_3 \qquad \mathtt{q(T_1)} \qquad \mathtt{q(T_2)}}{\Gamma_1 \vdash \mathtt{q} \ \mathtt{<t_1,t_2>} : \mathtt{q\ (T_1*T_2)}; \Gamma_3} \qquad \text{(A-Pair)}$$

$$\frac{\Gamma_1 \vdash \mathtt{t_1} : \mathtt{q\ (T_1*T_2)}; \Gamma_2 \qquad \Gamma_2, \mathtt{x{:}T_1}, \mathtt{y{:}T_2} \vdash \mathtt{t_2} : \mathtt{T}; \Gamma_3}{\Gamma_1 \vdash \mathtt{split\ t_1\ as\ x,y\ in\ t_2} : \mathtt{T}; \Gamma_3 \div (\mathtt{x{:}T_1}, \mathtt{y{:}T_2})} \qquad \text{(A-Split)}$$

$$\frac{\mathtt{q{=}un} \Rightarrow \Gamma_1 = \Gamma_2 \div (\mathtt{x{:}T_1}) \qquad \Gamma_1, \mathtt{x{:}T_1} \vdash \mathtt{t_2} : \mathtt{T_2}; \Gamma_2}{\Gamma_1 \vdash \mathtt{q}\ \lambda\mathtt{x{:}T_1.t_2} : \mathtt{q\ T_1{\to}T_2}; \Gamma_2 \div (\mathtt{x{:}T_1})} \qquad \text{(A-Abs)}$$

$$\frac{\Gamma_1 \vdash \mathtt{t_1} : \mathtt{q\ T_{11}{\to}T_{12}}; \Gamma_2 \qquad \Gamma_2 \vdash \mathtt{t_2} : \mathtt{T_{11}}; \Gamma_3}{\Gamma_1 \vdash \mathtt{t_1\ t_2} : \mathtt{T_{12}}; \Gamma_3} \qquad \text{(A-App)}$$

$$\Gamma \div \varnothing = \Gamma$$

$$\frac{\Gamma_1 \div \Gamma_2 = \Gamma_3 \qquad (\mathtt{x{:}lin\ P}) \notin \Gamma_3}{\Gamma_1 \div (\Gamma_2, \mathtt{x{:}lin\ P}) = \Gamma_3}$$

$$\frac{\Gamma_1 \div \Gamma_2 = \Gamma_3 \qquad \Gamma_3 = \Gamma_4, \mathtt{x{:}un\ P}, \Gamma_5}{\Gamma_1 \div (\Gamma_2, \mathtt{x{:}un\ P}) = \Gamma_4, \Gamma_5}$$

Extra notation:

- $\Gamma_{in} \vdash t : T; \Gamma_{out}$
- Difference in context: $\Gamma_1 \div \Gamma_2 = \Gamma_3$

Completeness & Soundness

1.2.5 LEMMA [ALGORITHMIC MONOTONICITY]: If $\Gamma \vdash \mathtt{t} : \mathtt{T};\Gamma'$ then $\mathcal{U}(\Gamma') = \mathcal{U}(\Gamma)$ and $\mathcal{L}(\Gamma') \subseteq \mathcal{L}(\Gamma)$.

1.2.6 LEMMA [ALGORITHMIC EXCHANGE]: If $\Gamma_1, \mathtt{x}_1{:}\mathtt{T}_1, \mathtt{x}_2{:}\mathtt{T}_2, \Gamma_2 \vdash \mathtt{t} : \mathtt{T};\Gamma_3$ then $\Gamma_1, \mathtt{x}_2{:}\mathtt{T}_2, \mathtt{x}_1{:}\mathtt{T}_1, \Gamma_2 \vdash \mathtt{t} : \mathtt{T};\Gamma_3'$ and Γ_3 is the same as Γ_3' up to transposition of the bindings for x_1 and x_2.

1.2.7 LEMMA [ALGORITHMIC WEAKENING]: If $\Gamma \vdash \mathtt{t} : \mathtt{T};\Gamma'$ then $\Gamma, \mathtt{x}{:}\mathtt{T}' \vdash \mathtt{t} : \mathtt{T};\Gamma', \mathtt{x}{:}\mathtt{T}'$.

1.2.8 LEMMA [ALGORITHMIC LINEAR STRENGTHENING]: If $\Gamma, \mathtt{x}{:}\mathtt{lin}\,\mathsf{P} \vdash \mathtt{t} : \mathtt{T};\Gamma', \mathtt{x}{:}\mathtt{lin}\,\mathsf{P}$ then $\Gamma \vdash \mathtt{t} : \mathtt{T};\Gamma'$.

1.2.9 THEOREM [ALGORITHMIC SOUNDNESS]: If $\Gamma_1 \vdash \mathtt{t} : \mathtt{T};\Gamma_2$ and $\mathcal{L}(\Gamma_2) = \varnothing$ then $\Gamma_1 \vdash \mathtt{t} : \mathtt{T}$.

1.2.10 THEOREM [ALGORITHMIC COMPLETENESS]: If $\Gamma_1 \vdash \mathtt{t} : \mathtt{T}$ then $\Gamma_1 \vdash \mathtt{t} : \mathtt{T};\Gamma_2$ and $\mathcal{L}(\Gamma_2) = \varnothing$.

Are there any questions?