Course TT & Log Exercises on Normalization

Some answers

1. Recall that $\sigma_1 \to \sigma_2 \cdots \sigma_n \to \alpha$ should be read as

$$\sigma_1 \to (\sigma_2 \to (\cdots \to (\sigma_n \to \alpha) \cdots))$$

and this means that every type $\sigma \to \tau$ can be written as

$$\sigma_1 \to \tau_1 \to \cdots \to \tau_n \to \alpha$$

where $\tau = \tau_1 \to \cdots \to \alpha$
by decomposing τ as far as possible.

We have two definitions of h:

$$h_1(\alpha) = 0 \qquad h_1(\sigma_1 \to \cdots \to \sigma_n \to \alpha) = \max(h_1(\sigma_1) \ldots h_1(\sigma_n)) + 1$$

$$h_2(\alpha) = 0 \qquad h_2(\sigma \to \tau) = \max(h_2(\sigma) + 1, h_2(\tau))$$

We want to show $h_1(\sigma) = h_2(\sigma)$ $(\forall \sigma \in \text{type})$
which we do by induction on σ

- Case $\sigma = \alpha$ ✓
- Case $\sigma = \tau_1 \to \cdots \to \tau_n \to \alpha$

$$\begin{aligned} h_2(\sigma) &= \max(h_2(\tau_1) + 1, h_2(\tau_2 \to \cdots \to \tau_n \to \alpha)) \\ &\overset{IH}{=} \max(h_1(\tau_1) + 1, h_1(\tau_2 \to \cdots \to \tau_n \to \alpha)) \\ &= \max(h_1(\tau_1) + 1, \max(h_1(\tau_1), \ldots h_1(\tau_n)) + 1) \\ &= \max(h_1(\tau_1), \ldots h_1(\tau_n)) + 1 \\ &= h_1(\sigma) \end{aligned}$$

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(2) In M there are two redexes:

* $I\,I$, whose height is the height of the type of the I on the left.
  Its type is C, so $h(II) = h(C)$
* the whole term M, whose height is the height of the type of $\lambda x: A.\, x(\lambda v: B.\, x\, I)$,
  which is $A \to B$, so the height is $h(A \to B)$

This means that the redex with greatest height is the whole term M, so we contract that redex.

NB
$$h(B) = h(\alpha \to \alpha) = h(\alpha) + 1 = 1$$
$$h(C) = h(B) + 1 = 2$$
$$h(A) = \max(h(C)+1, h(B)) = h(C)+1 = 3$$

For $m(M)$ we look at height of the redex of maximum height.
This is $h(A \to B) = h(A) + 1 = 4$.
So $m(M) = (4, 1)$

In N we have 4 redexes, 2 of height $h(C)$, the II ones, and 2 of height $h(C \to B)$, the ones with $\lambda z: C.\, z(II)$ as function

So we have 2 redexes of max height $h(C \to B) = 3$
So $m(N) = (3, 2)$

So $m(M) > m(N)$.

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3. If we define $f(M) :=$ the number of type-abstractions in M, then we have

$$M \xrightarrow[\beta]{} N \text{ using a type reduction}$$
$$\Rightarrow f(M) > f(N)$$

Definition by induction of f:

$$\begin{aligned}
f(\lambda\alpha.M) &= 1 + f(M)\\
f(\lambda x{:}\sigma.M) &= f(M)\\
f(M\,N) &= f(M) + f(N)\\
f(M\,\sigma) &= f(M)\\
f(x) &= 0
\end{aligned}$$

Then

$$\begin{tikzcd}
(\lambda\alpha.M)\sigma \arrow[r] \arrow[d, "f"] & M[\alpha:=\sigma] \arrow[d, "f"] \\
f(M)+1 & f(M)
\end{tikzcd}$$

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4. $X \in SAT$ if
   1. $x\vec{P} \in X$ for all $x \in Var$, $P_1 \dots P_n \in SN$
   2. $X \subseteq SN$
   3. If $M[x:=N]\vec{P} \in X$, ~~then~~ and $N \in SN$, then $(\lambda x.M)N\vec{P} \in X$.

Suppose $A, B \in SAT$

TP: $A \to B = \{M \mid \forall N \in A\ (MN \in B)\} \in SAT$

we check properties 1, 2, 3 for $A \to B$

1. let $x \in Var$, $P_1 \dots P_n \in SN$
   Then for $N \in A$ we know $N \in SN$, so $x P_1 \dots P_n N \in B$ (by $B \in SAT$), so $x P_1 \dots P_n \in A \to B$.
2. Let $M \in A \to B$. Take an element in A, say x (which is in A indeed, due to 1).
   So $Mx \in B$, so $Mx \in SN$, so $M \in SN$. (If $M \notin SN$, we would have an ∞ reduction from M, but then also from Mx)
3. let $M[x:=N]\vec{P} \in A \to B$ and let $Q \in A$ with $N \in SN$
   Then $M[x:=N]\vec{P}Q \in B$ by def of $A \to B$. So $(\lambda x.M)N\vec{P}Q \in B$, by prop 3 for B. This holds for all $Q \in A$, so $(\lambda x.M)N\vec{P} \in A \to B$

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5 Soundness Property

If $\Gamma \vdash M : \sigma$ where $\Gamma = x_1 : \tau_1 \ldots x_n : \tau_n$

then for all valuations ρ (so $\rho(\alpha_j) \in SAT$) and all $P_1 \ldots P_n$ with $P_i \in [\![\tau_i]\!]_\rho$

we have $M[x_1 := P_1 \ldots x_n := P_n] \in [\![\sigma]\!]_\rho$

Proof by induction on the derivation of $\Gamma \vdash M : \sigma$.

Case $\forall$-intro

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M : \forall \alpha . \sigma} \text{ if } \alpha \notin FV(\Gamma)$$

IH: $M[x_1 := P_1, \ldots, x_n := P_n] \in [\![\sigma]\!]_\rho$
for all ρ and $P_1 \ldots P_n$

Let ρ be a valuation, and $P_1 \ldots P_n$ with $P_i \in [\![\tau_i]\!]_\rho$

Then $M[x_1 := P_1 \ldots x_n := P_n] \in [\![\sigma]\!]_{\rho(\alpha = X)}$ for all $X \in SAT$,

because $[\![\tau_i]\!]_\rho$ only depends on ρ for the type variables $\beta \in FV(\tau_i)$

So $M[x_1 := P_1 \ldots x_n := P_n] \in \bigcap_{X \in SAT} [\![\sigma]\!]_{\rho(\alpha = X)} = [\![\forall \alpha . \sigma]\!]_\rho$

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