Classical axioms

The extra axioms in lean

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3th December 2021

Outline

Introdcution

Different views

- Constructive
  - Core parts of the library
- Classical
  - Extra Axioms

Implications

- Axiom of choice $\Longrightarrow$ Excluded middle
- Propositional Extensionality
- Quotients $\Longleftrightarrow$ Functional extensionality

Extra Axioms

Axiom of Choice:

$$\mathit{nonempty} := \lambda\alpha : U_u.\mu t : \mathbb{P}.(\mathit{intro}\ \alpha \to t)$$
$$\mathit{choice} : \forall\alpha : U_u.\mathit{nonempty}\ \alpha \to \alpha$$

Propositional Extensionality:

$$\forall p, q : \mathbb{P}\ ((p \to q \wedge q \to p) \to p = q)$$

Extra Axioms

Quotients:

$$\alpha/R : U_u$$
$$[\![\alpha]\!] : \alpha \to \alpha/R$$
$$sound_R : \forall x, y : \alpha.R\ x\ y \to [\![x]\!] = [\![y]\!]$$
$$lift_R : \forall \beta : U_v \forall f : \alpha \to \beta (\forall x, y : \alpha.R\ x\ y \to fx = fy) \to \alpha/R \to \beta$$
$$lift_R\ \beta\ f\ h\ [\![\alpha]\!] \rightsquigarrow f\ a$$

Extra Axioms

- η-rule

$$\frac{\Gamma \vdash e : \forall y : \alpha.\beta}{\Gamma \vdash \lambda x : \alpha.e\ x \equiv e}$$

- Proof irrelevance

$$\frac{\Gamma \vdash p : \mathbb{P} \quad \Gamma \vdash h : p \quad \Gamma \vdash h' : p}{\Gamma \vdash h \equiv h'}$$

Extra Axioms

- Law of excluded middle

$$\forall p : \mathbb{P}\ (p \vee \neg p)$$

- Functional Extensionality

$$\forall f_1, f_2\ (\forall x\ (f_1(x) = f_2(x)) \to f_1 = f_2)$$