Path types

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13/12/2022

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Introduction

Basis

Paper 'Cubical Type Theory: a constructive interpretation of the univalence axiom' by Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg.
Continuation of program started in 'A Model of Type Theory in Cubical Sets' by M. Bezem, T. Coquand, and S. Huber.

Goal

We provide a constructive justification of Voevodsky's univalence axiom, based on a nominal extension of λ-calculus, using names to formally represent elements of the unit interval [0, 1].

Introduction
Basis

Layout of the paper (9 sections):

1. Introduction (Now)

Introduction

Basis

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1. Introduction (Now)
2. Basic type theory (Now)

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3. Path Types (Now)

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4. Systems, composition, and transport (Next presentation)

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5. Derived notions and operations (Next presentation)

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2. Basic type theory (Now)
3. Path Types (Now)
4. Systems, composition, and transport (Next presentation)
5. Derived notions and operations (Next presentation)
6. Glueing (Tomorrow)

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7. Universe and the univalence axiom (Tomorrow)

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8. Semantics (Not presented)

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7. Universe and the univalence axiom (Tomorrow)
8. Semantics (Not presented)
9. Extensions: identity types and higher inductive types (Not presented)

Introduction
Refinement of semantics

We add new operations on names corresponding to the fact that the interval [0, 1] is canonically a de Morgan algebra.

- Simplify semantical justifications

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- Diagonal operation for HITs

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Refinement of semantics

We add new operations on names corresponding to the fact that the interval [0, 1] is canonically a de Morgan algebra.

- Simplify semantical justifications
- Diagonal operation for HITs
- Justify computation rule of Martin-Löf identity with simple definition

Introduction

Type system

Extension of Martin-Löf type theory by adding two new operations on contexts: Addition of new names representing dimensions and a restriction operation.

- Notion of extensibility generalizing notion of being connected via a path.

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Type system

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- Notion of extensibility generalizing notion of being connected via a path.
- Kan composition expressing the preservation of being extensible along paths.

Introduction

Type system

Extension of Martin-Löf type theory by adding two new operations on contexts: Addition of new names representing dimensions and a restriction operation.

- Notion of extensibility generalizing notion of being connected via a path.
- Kan composition expressing the preservation of being extensible along paths.
- Exensibility is preserved by equivalences using a new operation.

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Basic type theory

We use dependent type theory with a type for natural numbers, but without universes for now. We have β and η-conversion for dependent functions and surjective pairing for dependent pairs.

Syntax

Γ, Δ	$::=$	$() \mid \Gamma, x : A$	Contexts
t, u, A, B	$::=$	$x \mid \lambda x : A.\, t \mid t\ u \mid (x : A) \to B$	Π-types
	$\mid$	$(t, u) \mid t.1 \mid t.2 \mid (x : A) \times B$	Σ-types
	$\mid$	$0 \mid \mathsf{s}\ u \mid \mathsf{natrec}\ t\ u \mid \mathsf{N}$	Natural numbers

Basic type theory

The non-dependent function space is written as $A \to B$, and the type of non dependent pairs is written as $A \times B$. We consider terms and types up to α-equivalance of bound variables, and we define substitutions as $\sigma = (x_1/u_1, \ldots, x_n/u_n)$.

Basic type theory

We define $\Delta \vdash \sigma : \Gamma$ by induction on Γ.

Substitution

Base case: We have $\Delta \vdash () : ()$
IH: We have $\Delta \vdash (\sigma, x/u) : \Gamma, x : A$ if $\Delta \vdash \sigma : \Gamma$ and $\Delta \vdash u : A\sigma$

Basic type theory

Lemma 1

Substitution is admissible: $\dfrac{\Gamma \vdash J \qquad \Delta \vdash \sigma : \Gamma}{\Delta \vdash J\sigma}$

In particular, weakening is admissible: A judgment valid in a context stays valid in any extension of this context.

Basic type theory

Inference rules 1/5

(The condition $x \notin \mathrm{dom}(\Gamma)$ means that x is not declared in Γ)

$\boxed{\text{Well-formed contexts, } \Gamma \vdash}$

$$\frac{}{() \vdash} \qquad \frac{\Gamma \vdash A}{\Gamma, x : A \vdash} \quad (x \notin \mathrm{dom}(\Gamma))$$

Inference rules 2/5

$\boxed{\text{Well-formed types, } \Gamma \vdash A}$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \times B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{N}}$$

Basic type theory

Inference rules 3/5

Well-typed terms, $\Gamma \vdash t : A$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\, t : (x : A) \to B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash x : A}\ (x : A \in \Gamma)$$

$$\frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(x/u)} \qquad \frac{\Gamma \vdash t : (x : A) \times B}{\Gamma \vdash t.1 : A} \qquad \frac{\Gamma \vdash t : (x : A) \times B}{\Gamma \vdash t.2 : B(x/t.1)}$$

$$\frac{\Gamma, x : A \vdash B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t, u) : (x : A) \times B} \qquad \frac{\Gamma \vdash}{\Gamma \vdash 0 : \mathsf{N}} \qquad \frac{\Gamma \vdash n : \mathsf{N}}{\Gamma \vdash \mathsf{s}\ n : \mathsf{N}}$$

$$\frac{\Gamma, x : \mathsf{N} \vdash P \qquad \Gamma \vdash a : P(x/0) \qquad \Gamma \vdash b : (n : \mathsf{N}) \to P(x/n) \to P(x/\mathsf{s}\ n)}{\Gamma \vdash \mathsf{natrec}\ a\ b : (x : \mathsf{N}) \to P}$$

Basic type theory

Inference rules 4/5

Type equality, $\Gamma \vdash A = B$ (Congruence and equivalence rules which are omitted)

Term equality, $\Gamma \vdash a = b : A$ (Congruence and equivalence rules are omitted)

$$\frac{\Gamma \vdash t = u : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t = u : B} \qquad \frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x : A.\ t)\ u = t(x/u) : B(x/u)}$$

$$\frac{\Gamma, x : A \vdash t\ x = u\ x : B}{\Gamma \vdash t = u : (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t, u).1 = t : A}$$

$$\frac{\Gamma, x : A \vdash B \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t, u).2 = u : B(x/t)}$$

Basic type theory

Inference rules 5/5

$$\frac{\Gamma, x : A \vdash B \qquad \Gamma \vdash t.1 = u.1 : A \qquad \Gamma \vdash t.2 = u.2 : B(x/t.1)}{\Gamma \vdash t = u : (x : A) \times B}$$

$$\frac{\Gamma, x : \mathsf{N} \vdash P \qquad \Gamma \vdash a : P(x/0) \qquad \Gamma \vdash b : (n : \mathsf{N}) \to P(x/n) \to P(x/\mathsf{s}\ n)}{\Gamma \vdash \mathsf{natrec}\ a\ b\ 0 = a : P(x/0)}$$

$$\frac{\Gamma, x : \mathsf{N} \vdash P \qquad \Gamma \vdash a : P(x/0) \qquad \Gamma \vdash b : (n : \mathsf{N}) \to P(x/n) \to P(x/\mathsf{s}\ n) \qquad \Gamma \vdash n : \mathsf{N}}{\Gamma \vdash \mathsf{natrec}\ a\ b\ (\mathsf{s}\ n) = b\ n\ (\mathsf{natrec}\ a\ b\ n) : P(x/\mathsf{s}\ n)}$$

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Path types

$\mathbb{I}$: free de Morgan algebra on discrete, infinite set $i, j, k, \ldots$ representing directions.

$\mathbb{I}$ is a bounded, distributive lattice with top element 1 and bottom element 0 with an involution $1 - r$.

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$\mathbb{I}$ is a bounded, distributive lattice with top element 1 and bottom element 0 with an involution $1 - r$.

- $r, s \ ::= \ 0 \mid 1 \mid i \mid 1 - r \mid r \wedge s \mid r \vee s$
- $1 - 0 = 1$
- $1 - 1 = 0$
- $1 - (r \vee s) = (1 - r) \wedge (1 - s)$
- $1 - (r \wedge s) = (1 - r) \vee (1 - s)$

Path types

$\mathbb{I}$ has decibale equility and can be described as the free distributive lattice generated by symbols i and $1 - i$. Elements in $\mathbb{I}$ are formal representations of elements in [0, 1].

$$r \wedge s = \min(r, s)$$
$$r \vee s = \max(r, s)$$

Path types: context syntax

Contexts:

$$\Gamma, \Delta \ ::= \ \ldots \mid \Gamma, i : \mathbb{I}$$

$$\frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \ (i \notin \mathrm{dom}(\Gamma))$$

Path types: Type Theory syntax

Dependent Type Theory:

$$r, u, A, B \ ::= \ \ldots \mid \mathrm{Path}\ A\ t\ u \mid \langle i \rangle t \mid t\ r$$

Aside: notation

Special substitution cases

$$
\begin{aligned}
t(i/0) &\equiv t(i0)\\
t(i/1) &\equiv t(i1)
\end{aligned}
$$

Aside: notation

Special substitution cases

$$
\begin{aligned}
t[i ::= 0] &\equiv t(i/0) \equiv t(i0) \\
t[i ::= 1] &\equiv t(i/1) \equiv t(i1)
\end{aligned}
$$

Path types: inference rules (1/3)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathrm{Path}\ A\ t\ u}$$

Path types: inference rules (1/3)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathrm{Path}\ A\ t\ u}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathrm{Path}\ A\ t\ u}$$

Path types: inference rules (1/3)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash \text{Path } A\ t\ u}$$

$$\frac{\Gamma \vdash A \quad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \text{Path } A\ t\ u}$$

$$\frac{\Gamma, i : I \vdash t\ i = u\ i : A}{\Gamma \vdash t = u : \text{Path } A\ u_0\ u_1}$$

Path types: inference rules (2/3)

$$\frac{\Gamma \vdash t : \mathrm{Path}\ A\ u_0\ u_1 \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\, r : A}$$

Path types: inference rules (2/3)

$$\frac{\Gamma \vdash t : \mathrm{Path}\ A\ u_0\ u_1 \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\, r : A}$$

$$\frac{\Gamma \vdash t : \mathrm{Path}\ A\ u_0\ u_1}{\Gamma \vdash t\, 0 = u_0 : A} \qquad \frac{\Gamma \vdash t : \mathrm{Path}\ A\ u_0\ u_1}{\Gamma \vdash t\, 1 = u_1 : A}$$

Path types: inference rules (3/3)

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\langle i \rangle\, t)\, r = t(i/r) : A}$$

Path types: inference rules (3/3)

$$1_a = \langle i \rangle a$$

$$1_a : \mathrm{Path}\ A\ a\ a$$

Picture time!

Examples (1/3)

Equal elements → equal images

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash f : A \to B \qquad \Gamma \vdash p : \text{Path } A\ a\ b}{\Gamma \vdash \langle i \rangle f\ (p\ i) : \text{Path } B\ (f\ a)\ (f\ b)}$$

Examples (1/3)

Equal elements → equal images

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash f : A \to B \qquad \Gamma \vdash p : \text{Path } A\ a\ b}{\Gamma \vdash \langle i \rangle f\,(p\,i) : \text{Path } B\ (f\ a)\ (f\ b)}$$

$i/0$: $\quad ((\langle i \rangle f\,(p\,i))\ 0 = f\,(p\,0)) = f\ a$

Examples (1/3)

Equal elements → equal images

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash f : A \to B \qquad \Gamma \vdash p : \text{Path } A \ a \ b}{\Gamma \vdash \langle i \rangle f\,(p\,i) : \text{Path } B \ (f\ a)\ (f\ b)}$$

$i/0$: $\quad ((\langle i \rangle f\,(p\,i))\ 0 = f\,(p\,0)) = f\ a$

$i/1$: $\quad ((\langle i \rangle f\,(p\,i))\ 1 = f\,(p\,1)) = f\ b$

Examples (2/3)

Function extensionality

Let $j_0 = \Gamma \vdash f : (x : A) \to B$ and let $j_1 = \Gamma \vdash g : (x : A) \to B$. Then

$$\frac{j_0 \qquad j_1 \qquad \Gamma \vdash p : (x : A) \to \text{Path } B\ (f\,x)\ (g\,x)}{\Gamma \vdash \langle i \rangle \lambda x : A.p\,x\,i : \text{Path } ((x : A) \to B)\ f\ g}$$

Examples (2/3)

Function extensionality

Let $j_0 = \Gamma \vdash f : (x : A) \to B$ and let $j_1 = \Gamma \vdash g : (x : A) \to B$. Then

$$\frac{j_0 \qquad j_1 \qquad \Gamma \vdash p : (x : A) \to \text{Path } B\ (f\,x)\ (g\,x)}{\Gamma \vdash \langle i \rangle \lambda x : A.p\,x\,i : \text{Path } ((x : A) \to B)\ f\ g}$$

$i/0$: $\quad (\langle i \rangle \lambda x : A.p\,x\,i)\ 0 = \lambda x : A.p\,x\,0 = \lambda x : A.f\,x = f$

Examples (2/3)

Function extensionality

Let $j_0 = \Gamma \vdash f : (x : A) \to B$ and let $j_1 = \Gamma \vdash g : (x : A) \to B$. Then

$$\frac{j_0 \qquad j_1 \qquad \Gamma \vdash p : (x : A) \to \text{Path } B\ (f\,x)\ (g\,x)}{\Gamma \vdash \langle i \rangle \lambda x : A.p\,x\,i : \text{Path } ((x : A) \to B)\ f\ g}$$

$i/0$: $\quad (\langle i \rangle \lambda x : A.p\,x\,i)\ 0 = \lambda x : A.p\,x\,0 = \lambda x : A.f\,x = f$

$i/1$: $\quad (\langle i \rangle \lambda x : A.p\,x\,i)\ 1 = \lambda x : A.p\,x\,1 = \lambda x : A.g\,x = g$

Examples 3/3

Singletons are contractible

Any element in $(x : A) \times (\mathrm{Path}\ A\ a\ x)$ is equal to $(a, 1_a)$:

$$\frac{\Gamma \vdash p : \mathrm{Path}\ A\ a\ b}{\Gamma \vdash \langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)) : \mathrm{Path}\ (x : A) \times (\mathrm{Path}\ A\ a\ x)\ (a, 1_a)\ (b, p)}$$

Examples 3/3

Singletons are contractible

$$\frac{\Gamma \vdash p : \mathrm{Path}\ A\ a\ b}{\Gamma \vdash \langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)) : \mathrm{Path}\ (x : A) \times (\mathrm{Path}\ A\ a\ x)\ (a, 1_a)\ (b, p)}$$

Examples 3/3

Singletons are contractible

$$\frac{\Gamma \vdash p : \text{Path } A\ a\ b}{\Gamma \vdash \langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)) : \text{Path } (x : A) \times (\text{Path } A\ a\ x)\ (a, 1_a)\ (b, p)}$$

$i/0$:

$$\begin{aligned}
&(\langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)))\ 0 = \\
&(p\, 0, \langle j \rangle p\, (0 \wedge j)) = (p\, 0, \langle j \rangle p\, 0) = \\
&(a, \langle j \rangle a) = (a, 1_a)
\end{aligned}$$

Examples 3/3

Singletons are contractible

$$\frac{\Gamma \vdash p : \text{Path } A\ a\ b}{\Gamma \vdash \langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)) : \text{Path } (x : A) \times (\text{Path } A\ a\ x)\ (a, 1_a)\ (b, p)}$$

$i/0$:

$$(\langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)))\ 0 =$$
$$(p\, 0, \langle j \rangle p\, (0 \wedge j)) = (p\, 0, \langle j \rangle p\, 0) =$$
$$(a, \langle j \rangle a) = (a, 1_a)$$

$i/1$:

$$(\langle i \rangle (p\, i, \langle j \rangle p\, (i \wedge j)))\ 1 =$$
$$(p\, 1, \langle j \rangle p\, (1 \wedge j)) = (p\, 1, \langle j \rangle p\, j) =$$
$$(b, \langle j \rangle p\, j) = (b, p)$$