HoTT - Operations on Paths

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Introduction

Paths are really cool. But we can't even concatenate them yet...
Time to do this, and more.

Face lattice

We will be using the *face lattice* $\mathbb{F}$, described by the following grammar:

$$\phi, \psi ::= 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \phi \wedge \psi \mid \phi \vee \psi$$

Geometrically, these are faces of a cube. We use these to restrict our path types from the distributive lattice. This means we can now talk about parts of a cube, restricted contexts.

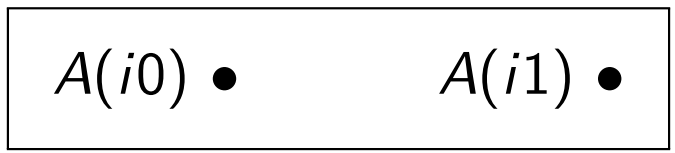
Face lattice

Given a face formula we can restrict contexts:

$$\frac{\Gamma \vdash \phi : \mathbb{F}}{\Gamma, \phi \vdash}$$

(Recall: this means Γ, ϕ is a valid context).

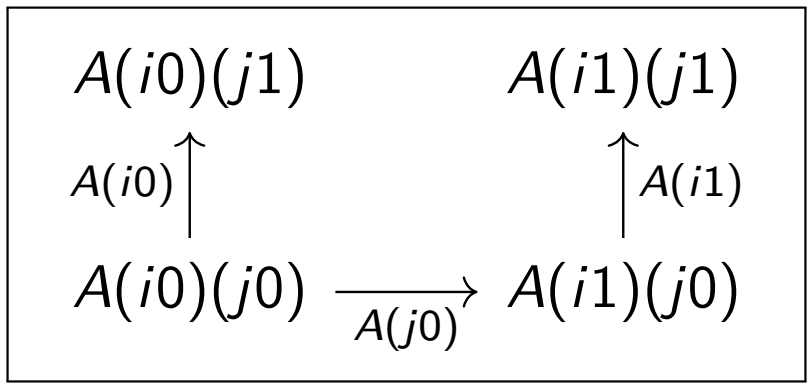
- $i : \mathbb{I}, 1_{\mathbb{F}} \vdash A$
  - simply a type.
- $i : \mathbb{I}, (i = 0) \vee (i = 1) \vdash A$
  - two unrelated types:



Partial cubes

If $i : \mathbb{I} \vdash A$ is a path, then

- $i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vee (i = 1) \vee (j = 0) \vdash A$:

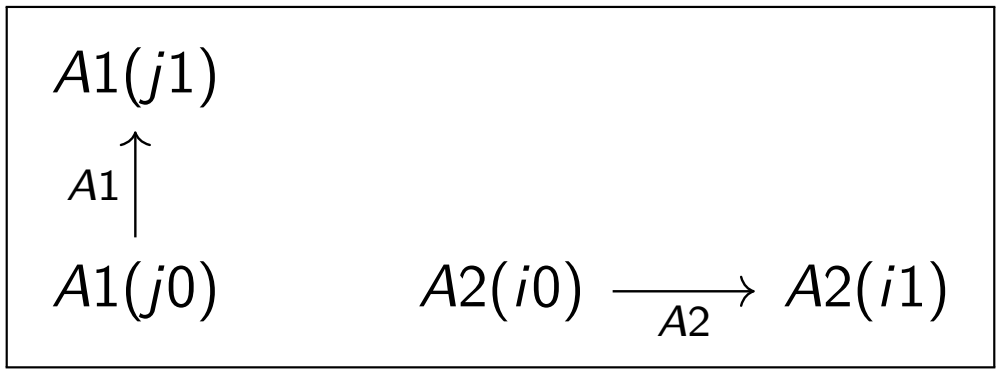


is a pot without a lid.

Systems

What if we have a bunch of sides

- $i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vdash A_1$
- $i : \mathbb{I}, j : \mathbb{I}, (j = 0) \vdash A_2$
- ...



but no square?
We don't have a problem, we have a system!

$$i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vee (j = 0) \vdash [(i = 0)A_1, (j = 0)A_2]$$

Systems

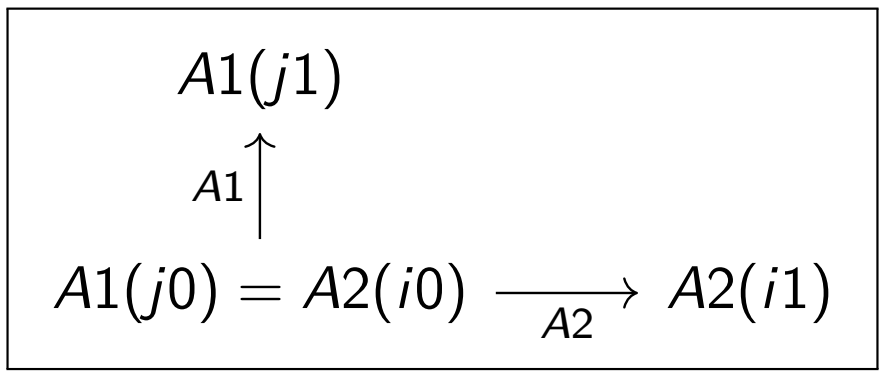
The system

$$\Gamma \vdash [\phi_1 A_1, \ldots]$$

is subject to conditions:

- covering (w.r.t. the context): $\Gamma \vdash \phi_1 \vee \cdots = 1_{\mathbb{F}}$
- compatibility: $\Gamma, \phi_i \wedge \phi_j \vdash A_i = A_j$

In the example we need:



Extensions

We very often work with

- restricted systems
- overlaps with total elements

So if $\Gamma, \phi \vdash u : A$, we abbreviate

$$\Gamma \vdash a : A, \Gamma, \phi \vdash a = u$$

to

$$\Gamma \vdash a : A[\phi \to u]$$

In other words, u is an extension, or, partial element, defined on ϕ.

Composition

$$t, u, A, B ::= \ldots \mid \mathrm{comp}^i\ A\ [\phi \mapsto u]\ a_0$$

Composition expresses that being extensible is preserved along paths.

$$\frac{\Gamma \vdash \phi : \mathbb{F} \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \phi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\phi \mapsto u(i0)]}{\Gamma \vdash \mathrm{comp}^i\ A\ [\phi \mapsto u] a_0 : A(i1)[\phi \mapsto u(i1)]}$$

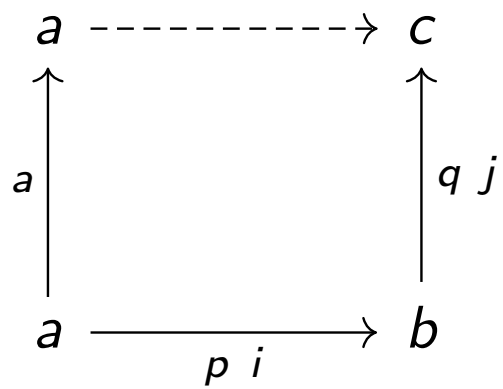
With the following equality judgment for systems:

$$\Gamma \vdash \mathrm{comp}^i\ A\ [1_{\mathbb{F}} \mapsto u]\ a_0 = u(i1) : A(i1)$$

Composition

Using composition we can justify transitivity of path types:

$$\frac{\Gamma \vdash p : \mathsf{Path}\ A\ a\ b \qquad \Gamma \vdash p : \mathsf{Path}\ A\ b\ c}{\Gamma \vdash \langle i \rangle\ \mathsf{comp}^j\ A\ [(i = 0) \mapsto a, (i = 1) \mapsto q\ j](p\ i) : \mathsf{Path}\ A\ a\ c}$$



Filling

Using composition we can also define a Kan filling operation.

$\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i\ A\ [\phi \mapsto u]\ a_0 = \mathsf{comp}^j\ A(i/i \wedge j)\ [\phi \mapsto u(i/i \wedge j), (i = 0) \mapsto a_0]\ a_0 : A$

Where $\Gamma, i : \mathbb{I} \vdash v = \mathsf{fill}^i\ A\ [\phi \mapsto u]\ a_0 : A$ satisfies

$$\Gamma \vdash v(i0) = a_0 : A(i0)$$

$$\Gamma \vdash v(i1) = \mathsf{comp}^i\ A\ [\phi \mapsto u]a_0 : A(i1)$$

$$\Gamma, \phi, i : \mathbb{I} \vdash v = u : A$$

Now we can compute the lid and filling of our pots.

Operations: transport

Transport corresponds to composition for $\phi = 0_{\mathbb{F}}$

$$\Gamma \vdash \mathsf{transp}^i\ A\ a = \mathsf{comp}^i\ A\ []\ a : A(i1)$$

so that

$$\Gamma \vdash \mathsf{transp}^i\ A : A(i0) \to A(i1)$$

Operations: contractible

We define isContr $A = (x : A) \times ((y : A) \to \mathrm{Path}\ A\ x\ y)$. Now given $\Gamma \vdash p : \mathrm{isContr}\ A$ and $\Gamma, \phi \vdash u$ we define the operation:

$$\Gamma \vdash \mathrm{contr}\ p\ [\phi \mapsto u] = \mathrm{comp}^i\ A\ [\phi \mapsto p.2\ u\ i]\ p.1 : A[\phi \mapsto u]$$

Lemma 5

Assume we have one operation

$$\frac{\Gamma, \phi \vdash u : A}{\Gamma \vdash \mathrm{contr}\ [\phi \mapsto u] : A[\phi \mapsto u]}$$

then we can find an element in isContr A.

Operations: contractible

Proof

We define $x = \text{contr } [] : A$ and prove that any $y : A$ is path equal to x. Say

$$\phi = (i = 0) \vee (i = 1) \qquad u = [(i = 0) \mapsto x, (i = 1) \mapsto y]$$

Then we obtain $\Gamma, i : \mathbb{I} \vdash v = \text{contr } [\phi \mapsto u] : A[\phi \mapsto u]$. In this way we get a path connecting x and y.

Operations: preserves

The pres operation states that any function preserves composition, up to path equality.

Lemma 6

We have an operation

$$\frac{\Gamma, i : \mathbb{I} \vdash f : T \to A \qquad \Gamma \vdash \phi : \mathbb{F} \qquad \Gamma, \phi, i : \mathbb{I} \vdash t : T \qquad \Gamma \vdash t_0 : T(i0)[\phi \mapsto t(i0)}{\Gamma \vdash \mathsf{pres}^i\ f\ [\phi \mapsto t]\ t_0 : (\mathsf{Path}\ A(i1)\ c_1\ c_2)[\phi \mapsto \langle j \rangle (f\ t)(i1)]}$$

where

$$c_1 = \mathsf{comp}^i\ A\ [\phi \mapsto f\ t](f(i0)\ t0)$$
$$c_2 = f(i1)(\mathsf{comp}^i\ T\ [\phi \mapsto t]\ t_0)$$

Operations: preserves

Proof

Let

$$\Gamma \vdash a_0 = f\,(i0)\ \ t0 : A(i0)$$
$$\Gamma, i : \mathbb{I} \vdash v = \mathsf{fill}^i\ \ T\ \ [\phi \mapsto t]\ \ t_0 : T$$

We take

$$\mathsf{pres}^i\ \ f\ \ [\phi \mapsto t]\ \ t_0 = \langle j \rangle\ \mathsf{comp}^i\ \ A\ \ [\phi \vee (j = 1) \mapsto f\ \ v]\ \ a_0$$

Operations: equivalence

We define:

$$\mathsf{isEquiv}\, T\ A\ f = (y : a) \to \mathsf{isContr}\ ((x : T) \times \mathsf{Path}\ A\ y\ (f\ x))$$
$$\mathsf{Equiv}\ T\ A = (f : T \to A) \times \mathsf{isEquiv}\ T\ A\ f$$

Lemma 7

If $\Gamma \vdash f : \mathsf{Equiv}\ T\ A$, we have an operation

$$\frac{\Gamma, \phi \vdash t : T \qquad \Gamma \vdash a : A \qquad \Gamma, \phi \vdash p : \mathsf{Path}\ A\ a\ (f\ t)}{\Gamma \vdash \mathsf{equiv}\ f\ [\phi \mapsto (t, p)]a : ((x : T) \times \mathsf{Path}\ A\ a\ (f\ x))[\phi \mapsto (t, p)]}$$

Conversely, if we have such an operation, then we can build a proof that f is an equivalence.

Proof sketch

A theorem states that contractible maps are equivalences. We use our definition of contractible to define
equiv $f\ [\phi \mapsto (t, p)]\ a = \text{contr}\ (f.2\ a)\ [\phi \mapsto (t, p)]$.

Computing composition

We can define the equality judgments for composition, but particularly product and sum types are rather technical.

Computing composition: Natural Numbers

For $C = N$ we define $\mathrm{comp}^i\ C\ [\phi \mapsto n]\ n_0$ by recursion:

$$\Gamma \vdash \mathrm{comp}^i\, C\ [\phi \mapsto 0]\ 0 = 0 : C$$

$$\Gamma \vdash \mathrm{comp}^i\, C\ [\phi \mapsto s\ n](s\ n_0) = s\ (\mathrm{comp}^i\, C\ [\phi \mapsto n] n_0) : C$$

Computing composition: Path Types

Given

$$C = \mathsf{Path}\ A\ u\ v$$
$$\Gamma, \phi, i : \mathbb{I} \vdash p : C$$
$$\Gamma \vdash p_0 : C(i0)[\phi \mapsto p(i0)]$$

we define

$\Gamma \vdash \mathsf{comp}^i\ C\ [\phi \mapsto p]\ p_0 = \langle j \rangle\ \mathsf{comp}^i\ A\ [\phi \mapsto p\ j, (j = 0) \mapsto u, (j = 1) \mapsto v]\ (p_0, j) : C(i1)$

Computing composition: Product Types

For some $C = (x : A) \to B$ we set

$$w = \mathsf{fill}^i A(i/1 - i)[]u_1$$
$$v = w(i/1 - i)$$

Defining:

$\Gamma \vdash (\mathsf{comp}^i C[\phi \mapsto \mu]\lambda_0)u_1 = \mathsf{comp}^i B(x/v)[\phi \mapsto \mu v](\lambda_0 v(i0)) : B(x/v)(i1)$

Computing composition: Sum Types

Given

$$C = (x : A) \times B$$
$$\Gamma, \phi, i : \mathbb{I} \vdash w : C$$
$$\Gamma \vdash w_0 : C(i0)[\phi \mapsto w(i0)]$$

we let

$$a = \mathsf{fill}^i\ A\ [\phi \mapsto w.1]\ w_0.1$$
$$c_1 = \mathsf{comp}^i\ A\ [\phi \mapsto w.1]\ w_0.1$$
$$c_2 = \mathsf{comp}^i\ B(x/a)\ [\phi \mapsto w.2]\ w_0.2$$

Defining:

$$\Gamma \vdash \mathsf{comp}C[\phi \mapsto w]\ w_0 = (c1, c2) : C(i1)$$