

propositional logic & simple types

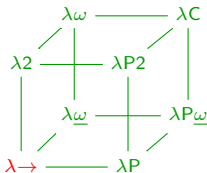
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Type Theory & Coq

2023–2024

Radboud University Nijmegen

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type systems and logics

teaser: a proof term for a proof

$\lambda x : a \rightarrow b \rightarrow c. \lambda y : a \rightarrow b. \lambda z : a. xz(yz)$

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad [a^z]}{b \rightarrow c} E \rightarrow \quad \frac{[a \rightarrow b^y] \quad [a^z]}{b} E \rightarrow}{c} I[z] \rightarrow}{\frac{(a \rightarrow b) \rightarrow a \rightarrow c}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow} I[x] \rightarrow$$

many systems

- ▶ one untyped lambda calculus
(last week: recap)
- ▶ many typed lambda calculi = type theories
 - ▶ **Church-style**
variables explicitly typed

the systems in this course
Coq!
 - ▶ **Curry-style**
assigning types to untyped terms
(in one of Herman's lectures)
- ▶ many logics

Curry-Howard correspondence

'propositions-as-types'

type systems \sim logics

datatypes \sim propositions

$A \times B$ \sim $A \wedge B$

$A \rightarrow B$ \sim $A \rightarrow B$

objects of a datatype \sim proofs of a proposition
'inhabitants' \sim 'proof objects'

non-empty type \sim true proposition

empty type \sim false proposition

Curry-Howard correspondence

'propositions-as-types'

type systems \sim logics

datatypes \sim propositions

$A \times B$ \sim $A \wedge B$

$A \rightarrow B$ \sim $A \rightarrow B$

objects of a datatype \sim proofs of a proposition
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non-empty type \sim true proposition

empty type \sim false proposition

derivation in a type theory \sim derivation in a logic

type systems in this course

logics \sim type systems

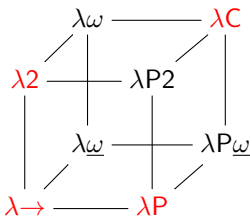
propositional logic \sim $\lambda \rightarrow$ = simply typed lambda calculus

predicate logic \sim λP = dependently typed lambda calculus

second order logic \sim $\lambda 2$ = polymorphic lambda calculus

higher order logic \sim λC = CC = Calculus of Constructions

'the logic of Coq' \sim CIC = Calculus of Inductive Constructions



type systems in this course

logics \sim type systems

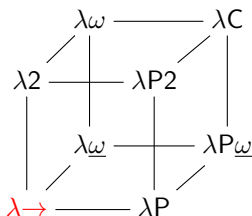
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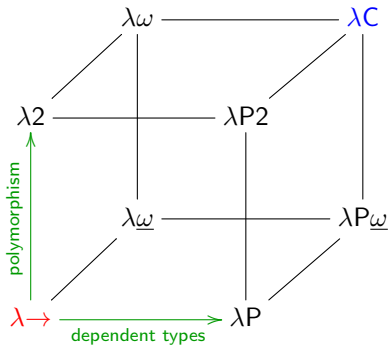


the lambda cube

the Barendregt cube

eight PTSs = pure type systems

(not explicitly in Femke's course notes)



▶ **natural deduction**

$$B_1, \dots, B_m \vdash A$$

introduction and elimination rules for each connective

$$\frac{\dots \vdash \dots}{\dots \vdash \dots \otimes \dots} I$$

$$\frac{\dots \vdash \dots \otimes \dots}{\dots \vdash \dots} E$$

- ▶ Gentzen–Prawitz style: proof is a derivation tree
- ▶ Jaśkowski–Fitch style: proof consists of nested boxes/flags

▶ **sequent calculus**: LK & LJ

$$B_1, \dots, B_m \vdash A_1, \dots, A_n$$

left and right rules for each connective

$$\frac{\dots \vdash \dots}{\dots \otimes \dots \vdash \dots} L$$

$$\frac{\dots \vdash \dots}{\dots \vdash \dots \otimes \dots} R$$

▶ **Hilbert-style**

$$\vdash A$$

axioms for each connective (and at most two rules)

- ▶ **natural deduction** $B_1, \dots, B_m \vdash A$
introduction and elimination rules for each connective

$$\frac{\dots \vdash \dots}{\dots \vdash \dots \otimes \dots} I \qquad \frac{\dots \vdash \dots \otimes \dots}{\dots \vdash \dots} E$$

- ▶ **Gentzen–Prawitz style:** proof is a derivation tree
 - ▶ **Jaśkowski–Fitch style:** proof consists of nested boxes/flags
- ▶ **sequent calculus:** LK & LJ $B_1, \dots, B_m \vdash A_1, \dots, A_n$
left and right rules for each connective

$$\frac{\dots \vdash \dots}{\dots \otimes \dots \vdash \dots} L \qquad \frac{\dots \vdash \dots}{\dots \vdash \dots \otimes \dots} R$$

- ▶ **Hilbert-style** $\vdash A$
axioms for each connective (and at most two rules)

defining a logic or type system

- ▶ **syntax**

- ▶ terms
- ▶ types
- ▶ formulas
- ▶ contexts
- ▶ judgments

- ▶ **rules**

- ▶ **logics**: proof rules
- ▶ **type systems**: typing rules

- ▶ **reduction**

defining a logic or type system

▶ syntax

- ▶ terms
- ▶ types
- ▶ formulas
- ▶ contexts
- ▶ judgments

$$M, N ::= x \mid MN \mid \lambda x. M$$

▶ rules

- ▶ **logics**: proof rules
- ▶ **type systems**: typing rules

no rules for untyped lambda calculus

▶ reduction

$$(\lambda x. M) N \rightarrow_{\beta} M[x := N]$$

propositional logic

three propositional logics

▶ **minimal propositional logic**

the logic that corresponds to simply typed lambda calculus

only implication: \rightarrow

▶ **constructive propositional logic**

all connectives: $\rightarrow, \wedge, \vee, \neg, \perp, \top$

▶ **classical propositional logic**

$$A \vee \neg A$$

$$\neg\neg A \rightarrow A$$

propositional logic: syntax

formulas

$$A, B ::= a \mid A \rightarrow B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \perp$$

a, b, c, \dots atomic propositions
later: propositional variables

A, B, C, \dots meta-variables for arbitrary formulas

implication associates to the right:

$$a \rightarrow b \rightarrow c \text{ means } a \rightarrow (b \rightarrow c)$$

order of binding strength: $\neg > \wedge > \vee > \rightarrow$

$$a \vee \neg b \wedge c \rightarrow d \text{ means } (a \vee ((\neg b) \wedge c)) \rightarrow d$$

the two rules of minimal propositional logic

the introduction and elimination rules of implication:

$$\frac{\begin{array}{c} [A^x] \\ \vdots \\ B \end{array}}{A \rightarrow B} I[x] \rightarrow \qquad \frac{\begin{array}{cc} \vdots & \vdots \\ A \rightarrow B & A \end{array}}{B} E \rightarrow$$

in the introduction rule the assumption $[A^x]$ may be used an arbitrary number of times: zero, one or more

in the elimination rule the proof of $A \rightarrow B$ is on the *left* of the proof of A

example proof of minimal propositional logic

$$(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x]$

$$\frac{(a \rightarrow b) \rightarrow a \rightarrow c}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow$$

example proof of minimal propositional logic

$$[a \rightarrow b \rightarrow c^x] \quad [a \rightarrow b^y]$$

$$\frac{\frac{a \rightarrow c}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x]$ $[a \rightarrow b^y]$ $[a^z]$

$$\frac{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x]$ $[a \rightarrow b^y]$ $[a^z]$

$$\frac{\frac{\frac{b \rightarrow c}{a \rightarrow c} I[z] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow \quad \frac{b}{E \rightarrow}$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x]$ $[a \rightarrow b^y]$ $[a^z]$

$$\frac{\frac{a \rightarrow b \rightarrow c \quad a}{b \rightarrow c} E \rightarrow \quad b}{E \rightarrow} E \rightarrow$$
$$\frac{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x] \quad [a \rightarrow b^y] \quad [a^z]$

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad a}{b \rightarrow c} E \rightarrow \quad b}{E \rightarrow} E \rightarrow}{\frac{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow \quad (a \rightarrow b) \rightarrow a \rightarrow c}{I[y] \rightarrow} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow} I[x] \rightarrow$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x] \quad [a \rightarrow b^y] \quad [a^z]$

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad [a^z]}{b \rightarrow c} E \rightarrow \quad b}{E \rightarrow}}{\frac{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow \quad (a \rightarrow b) \rightarrow a \rightarrow c}{I[y] \rightarrow}}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow}$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x]$ $[a \rightarrow b^y]$ $[a^z]$

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad [a^z]}{b \rightarrow c} E \rightarrow \quad \frac{a \rightarrow b \quad a}{b} E \rightarrow}{E \rightarrow}}{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow}$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x]$ $[a \rightarrow b^y]$ $[a^z]$

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad [a^z]}{b \rightarrow c} E \rightarrow \quad \frac{[a \rightarrow b^y] \quad a}{b} E \rightarrow}{E \rightarrow} \quad \frac{c}{a \rightarrow c} I[z] \rightarrow}{\frac{(a \rightarrow b) \rightarrow a \rightarrow c}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow} I[x] \rightarrow$$

example proof of minimal propositional logic

$[a \rightarrow b \rightarrow c^x] \quad [a \rightarrow b^y] \quad [a^z]$

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad [a^z]}{b \rightarrow c} E \rightarrow \quad \frac{[a \rightarrow b^y] \quad [a^z]}{b} E \rightarrow}{E \rightarrow}}{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow}$$

example proof of minimal propositional logic

$$\frac{\frac{\frac{[a \rightarrow b \rightarrow c^x] \quad [a^z]}{b \rightarrow c} E \rightarrow \quad \frac{[a \rightarrow b^y] \quad [a^z]}{b} E \rightarrow}{E \rightarrow}}{\frac{\frac{\frac{c}{a \rightarrow c} I[z] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow c} I[y] \rightarrow}{(a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I[x] \rightarrow}}$$

constructive propositional logic: the other rules

$$\frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ B \end{array}}{A \wedge B} I\wedge \quad \frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{A} E\wedge \quad \frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{B} E\wedge$$

$$\frac{\begin{array}{c} \vdots \\ A \end{array}}{A \vee B} I\vee \quad \frac{\begin{array}{c} \vdots \\ B \end{array}}{A \vee B} I\vee \quad \frac{\begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} \vdots \\ A \rightarrow C \end{array} \quad \begin{array}{c} \vdots \\ B \rightarrow C \end{array}}{C} E\vee$$

$$\frac{\begin{array}{c} [A^x] \\ \vdots \\ \perp \end{array}}{\neg A} I\neg \quad \frac{\begin{array}{c} \vdots \\ \neg A \end{array} \quad \begin{array}{c} \vdots \\ A \end{array}}{\perp} E\neg \quad \frac{}{\top} I\top \quad \frac{\begin{array}{c} \vdots \\ \perp \end{array}}{A} E\perp$$

constructive propositional logic: the other rules

$$\frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ B \end{array}}{A \wedge B} I\wedge \quad \frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{A} E\wedge \quad \frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{B} E\wedge$$

$$\frac{\begin{array}{c} \vdots \\ A \end{array}}{A \vee B} I\vee \quad \frac{\begin{array}{c} \vdots \\ B \end{array}}{A \vee B} I\vee \quad \frac{\begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} \vdots \\ A \rightarrow C \end{array} \quad \begin{array}{c} \vdots \\ B \rightarrow C \end{array}}{C} E\vee$$

$$\frac{\begin{array}{c} [A^x] \\ \vdots \\ \perp \end{array}}{\neg A} I\neg \quad \frac{\begin{array}{c} \vdots \\ \neg A \end{array} \quad \begin{array}{c} \vdots \\ A \end{array}}{\perp} E\neg \quad \frac{}{\top} I\top \quad \frac{\begin{array}{c} \vdots \\ \perp \end{array}}{A} E\perp$$

$$\neg A := A \rightarrow \perp$$

variants of elimination rules

- ▶ what Coq's `elim` tactic implements for conjunction:

$$\frac{\begin{array}{c} \vdots \\ A \wedge B \end{array} \quad \begin{array}{c} \vdots \\ A \rightarrow B \rightarrow C \end{array}}{C} E\wedge$$

- ▶ often the disjunction elimination rule is:

$$\frac{\begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} [A^x] \\ \vdots \\ C \end{array} \quad \begin{array}{c} [B^y] \\ \vdots \\ C \end{array}}{C} E[x, y]\vee$$

not what Coq's `elim` tactic implements for disjunction

example proof beyond minimal propositional logic

$$a \vee b \rightarrow b \vee a$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$$\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$$\frac{[a \vee b^x] \quad a \rightarrow b \vee a \quad b \rightarrow b \vee a}{b \vee a} E\vee$$
$$\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $[a^y]$

$$\frac{[a \vee b^x] \quad \frac{b \vee a}{a \rightarrow b \vee a} I[y] \rightarrow \quad b \rightarrow b \vee a}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} E\vee$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $[a^y]$

$$\frac{\frac{\frac{a}{b \vee a} Ir\vee}{a \rightarrow b \vee a} I[y] \rightarrow \quad b \rightarrow b \vee a}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} E\vee$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $[a^y]$

$$\frac{\frac{\frac{[a^y]}{b \vee a} Ir\vee}{a \rightarrow b \vee a} I[y] \rightarrow \quad b \rightarrow b \vee a}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} E\vee$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$[b^z]$

$$\frac{\frac{[a \vee b^x] \quad \frac{\frac{[a^y]}{b \vee a} Ir\vee}{a \rightarrow b \vee a} I[y] \rightarrow}{b \vee a} E\vee}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} I[z] \rightarrow$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$[b^z]$

$$\frac{
 \frac{
 \frac{
 [a^y]
 }{b \vee a} Ir\vee
 }{a \rightarrow b \vee a} I[y] \rightarrow
 \quad
 \frac{
 \frac{
 b
 }{b \vee a} Il\vee
 }{b \rightarrow b \vee a} I[z] \rightarrow
 }{E\vee}
 }{
 \frac{
 b \vee a
 }{a \vee b \rightarrow b \vee a} I[x] \rightarrow
 }$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$[b^z]$

$$\frac{\frac{[a^y]}{b \vee a} Ir\vee \quad \frac{[b^z]}{b \vee a} Il\vee}{\frac{a \rightarrow b \vee a}{b \rightarrow b \vee a} I[y] \rightarrow \quad \frac{b \rightarrow b \vee a}{E\vee} I[z] \rightarrow} \frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow$$

example proof beyond minimal propositional logic

$$\frac{[a \vee b^x] \quad \frac{\frac{[a^y]}{b \vee a} Ir\vee \quad I[y] \rightarrow}{a \rightarrow b \vee a} \quad \frac{\frac{[b^z]}{b \vee a} Il\vee \quad I[z] \rightarrow}{b \rightarrow b \vee a}}{E\vee} \quad \frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow}{a \vee b \rightarrow b \vee a}$$

alternative style: explicit assumption lists

syntax

| | |
|--|------------------|
| $A, B ::= a \mid A \rightarrow B \mid \dots$ | formulas |
| $\Gamma ::= \cdot \mid \Gamma, A$ | assumption lists |
| $\mathcal{J} ::= \Gamma \vdash A$ | sequents |

we do not write the dot and the comma after the dot
still natural deduction, *not* sequent calculus

rules

$$\frac{}{\Gamma \vdash A} \text{ass} \quad \text{for } A \in \Gamma$$
$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} I \rightarrow \quad \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} E \rightarrow \quad \dots$$

the example proofs with explicit assumption lists

$\Gamma_1 := a \rightarrow b \rightarrow c, a \rightarrow b, a$

$$\begin{array}{c}
 \frac{}{\Gamma_1 \vdash a \rightarrow b \rightarrow c} \text{ass} \quad \frac{}{\Gamma_1 \vdash a} \text{ass} \quad \frac{}{\Gamma_1 \vdash a \rightarrow b} \text{ass} \quad \frac{}{\Gamma_1 \vdash a} \text{ass} \\
 \frac{}{\Gamma_1 \vdash b \rightarrow c} E \rightarrow \quad \frac{}{\Gamma_1 \vdash b} E \rightarrow \\
 \frac{}{\Gamma_1 \vdash c} E \rightarrow \\
 \frac{}{a \rightarrow b \rightarrow c, a \rightarrow b \vdash a \rightarrow c} I \rightarrow \\
 \frac{}{a \rightarrow b \rightarrow c \vdash (a \rightarrow b) \rightarrow a \rightarrow c} I \rightarrow \\
 \frac{}{\vdash (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow a \rightarrow c} I \rightarrow
 \end{array}$$

$$\begin{array}{c}
 \frac{}{a \vee b \vdash a \vee b} \text{ass} \quad \frac{}{a \vee b, a \vdash a} \text{ass} \quad \frac{}{a \vee b, b \vdash b} \text{ass} \\
 \frac{}{a \vee b \vdash a \rightarrow b \vee a} I \rightarrow \quad \frac{}{a \vee b, a \vdash b \vee a} Ir \vee \quad \frac{}{a \vee b, b \vdash b \vee a} Il \vee \\
 \frac{}{a \vee b \vdash b \vee a} I \rightarrow \quad \frac{}{a \vee b \vdash a \rightarrow b \vee a} I \rightarrow \quad \frac{}{a \vee b \vdash b \rightarrow b \vee a} I \rightarrow \\
 \frac{}{\vdash a \vee b \rightarrow b \vee a} E \vee \\
 \frac{}{\vdash a \vee b \rightarrow b \vee a} I \rightarrow
 \end{array}$$

simply typed lambda calculus

the S combinator

$$\lambda xyz. xz(yz)$$

simply typed lambda calculus

the S combinator

$$\lambda xyz. xz(yz)$$
$$(\lambda x. (\lambda y. (\lambda z. ((xz)(yz)))))$$

simply typed lambda calculus

the S combinator

untyped

typed: Curry-style

$$\lambda xyz. xz(yz)$$
$$(\lambda x. (\lambda y. (\lambda z. ((xz)(yz)))))$$

typed: Church-style

$$\lambda x : a \rightarrow b \rightarrow c. \lambda y : a \rightarrow b. \lambda z : a. xz(yz)$$

simply typed lambda calculus

the S combinator

untyped

typed: Curry-style

$$\lambda xyz. xz(yz)$$
$$(\lambda x. (\lambda y. (\lambda z. ((xz)(yz)))))$$

typed: Church-style

$$\lambda x : a \rightarrow b \rightarrow c. \lambda y : a \rightarrow b. \lambda z : a. xz(yz)$$

Coq syntax

```
fun x : a -> b -> c => fun y : a -> b => fun z : a =>
  x z (y z)
```

simply typed lambda calculus

the S combinator

untyped

typed: Curry-style

$$\lambda xyz. xz(yz)$$
$$(\lambda x. (\lambda y. (\lambda z. ((xz)(yz)))))$$

typed: Church-style

$$\lambda x : a \rightarrow b \rightarrow c. \lambda y : a \rightarrow b. \lambda z : a. xz(yz)$$

Coq syntax

```
fun x : a -> b -> c => fun y : a -> b => fun z : a =>
  x z (y z)
```

```
fun (x : a -> b -> c) (y : a -> b) (z : a) => x z (y z)
```

BHK interpretation

Brouwer–Heyting–Kolmogorov

~ Curry-Howard correspondence

constructive ‘meaning’ of the logical connectives

connection between proofs and lambda calculus

| | | | |
|----------------------------|---|--------------------------------|--------------------------------------|
| proof of $A \rightarrow B$ | = | function | from proofs of A to proofs of B |
| proof of $A \wedge B$ | = | pair | of a proof of A and a proof of B |
| proof of $A \vee B$ | = | either | a proof of A , or a proof of B |
| proof of $\neg A$ | = | proof of $A \rightarrow \perp$ | |
| proof of \top | | the object I | |
| proof of \perp | | <i>does not exist</i> | |

BHK interpretation

Brouwer–Heyting–Kolmogorov

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| proof of \top | | the object I | |
| proof of \perp | | does not exist | |

proof of $a \wedge b \rightarrow b \wedge a$ is

a function that inputs a pair $\langle x, y \rangle$, and returns $\langle y, x \rangle$
with x a proof of a and y a proof of b

$$\lambda p : a \wedge b. \langle \pi_2 p, \pi_1 p \rangle$$

different names for the same type theory

simply typed lambda calculus

||

simple type theory = STT

||

$\lambda \rightarrow$

3 typing rules

\nrightarrow

7 typing rules

same set of well-typed terms

simply typed lambda calculus: syntax and rules

syntax

| | |
|--|-----------|
| $A, B ::= a \mid A \rightarrow B$ | types |
| $M, N ::= x \mid MN \mid \lambda x : A. M$ | preterms |
| $\Gamma ::= \cdot \mid \Gamma, x : A$ | contexts |
| $\mathcal{J} ::= \Gamma \vdash M : A$ | judgments |

rules

$$\frac{}{\Gamma \vdash x : A} \quad \text{for } (x : A) \in \Gamma$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B}$$

$$\frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

well-typedness

$\Gamma \vdash M : A$ is derivable $\implies M$ is **well-typed**

well-typed preterms are called **terms**

example type derivation

$\lambda f : a \rightarrow b. \lambda x : a. fx$

example type derivation

$\lambda f : a \rightarrow b. \lambda x : a. fx$

example type derivation

$\lambda f : a \rightarrow b. \lambda x : a. fx$

$\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$

$\Gamma \vdash M : A$

example type derivation

$\lambda f : a \rightarrow b. \lambda x : a. fx$

$\frac{\cdot}{\Gamma} \vdash \underbrace{(\lambda f : a \rightarrow b. \lambda x : a. fx)}_M : \underbrace{(a \rightarrow b) \rightarrow a \rightarrow b}_A$

$\Gamma \vdash M : A$

example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$
$$\cdot \vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$
$$\cdot \vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

$$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$
$$\cdot \vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$
$$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

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· $\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$

$$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

$\lambda f : a \rightarrow b. \lambda x : a. fx$

$\frac{}{\Gamma \vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$

(Note: In the original image, the lambda expression and its type are annotated with red brackets and labels: x under the lambda binder, A under a , M under fx , A under $a \rightarrow b$, and B under $a \rightarrow b$.)

$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$

example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$

$$\frac{\frac{\Gamma \quad x \quad A \quad M \quad B}{\cdot, f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\Gamma \vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

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example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$

$$\frac{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

$$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$

$$\frac{\frac{f : a \rightarrow b, x : a \vdash fx : b}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

$$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

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$$\frac{}{\Gamma \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$

$$\frac{\frac{\frac{\Gamma}{f : a \rightarrow b, x : a \vdash f : a \rightarrow b} \quad \frac{F \quad A \quad B}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b, x : a \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}}$$

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example type derivation

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$$\frac{\frac{\frac{f : a \rightarrow b, x : a \vdash f : a \rightarrow b \quad f : a \rightarrow b, x : a \vdash x : a}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

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example type derivation

$$\lambda f : a \rightarrow b. \lambda x : a. fx$$

$$\frac{\frac{\frac{\frac{f : a \rightarrow b, x : a \vdash f : a \rightarrow b}{\quad} \quad \frac{f : a \rightarrow b, x : a \vdash x : a}{\quad}}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

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$$\frac{\frac{\Gamma \vdash x : A}{\quad} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B}}{\Gamma \vdash x : A} \quad \frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

proofs versus type derivations: type derivation

$$\frac{\frac{\frac{}{f : a \rightarrow b, x : a \vdash f : a \rightarrow b} \quad \frac{}{f : a \rightarrow b, x : a \vdash x : a}}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

proofs versus type derivations: type derivation

$$\frac{\frac{\frac{}{f : a \rightarrow b, x : a \vdash f : a \rightarrow b} \quad \frac{}{f : a \rightarrow b, x : a \vdash x : a}}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

proofs versus type derivations: type derivation

$$\frac{\frac{\frac{}{f : a \rightarrow b, x : a \vdash f : a \rightarrow b} \quad \frac{}{f : a \rightarrow b, x : a \vdash x : a}}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

$$\frac{\frac{\frac{}{a \rightarrow b, a \vdash a \rightarrow b} \quad \frac{}{a \rightarrow b, a \vdash a}}{a \rightarrow b, a \vdash b}}{a \rightarrow b \vdash a \rightarrow b}}{\vdash (a \rightarrow b) \rightarrow a \rightarrow b}$$

proofs versus type derivations: type derivation

$$\frac{\frac{\frac{}{f : a \rightarrow b, x : a \vdash f : a \rightarrow b} \quad \frac{}{f : a \rightarrow b, x : a \vdash x : a}}{f : a \rightarrow b, x : a \vdash fx : b}}{f : a \rightarrow b \vdash (\lambda x : a. fx) : a \rightarrow b}}{\vdash (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b}$$

$$\frac{\frac{\frac{}{a \rightarrow b, a \vdash a \rightarrow b} \text{ass} \quad \frac{}{a \rightarrow b, a \vdash a} \text{ass}}{a \rightarrow b, a \vdash b} E \rightarrow \quad \frac{a \rightarrow b, a \vdash b}{a \rightarrow b \vdash a \rightarrow b} I \rightarrow}{\vdash (a \rightarrow b) \rightarrow a \rightarrow b} I \rightarrow$$

proofs versus type derivations: proof

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$$\frac{\frac{\frac{\frac{\frac{}{a \rightarrow b, a \vdash a \rightarrow b} \text{ass}}{a \rightarrow b, a \vdash b} I \rightarrow}{a \rightarrow b \vdash a \rightarrow b} I \rightarrow}{\vdash (a \rightarrow b) \rightarrow a \rightarrow b} I \rightarrow}{\frac{\frac{}{a \rightarrow b, a \vdash a} \text{ass}}{a \rightarrow b, a \vdash b} E \rightarrow} \text{ass}}$$

proofs versus type derivations: proof term

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$\lambda f : a \rightarrow b. \lambda x : a. fx$

Curry-Howard correspondence

propositions

types

implications

\longleftrightarrow

function types

proof rules

proof terms

introduction rules

\longleftrightarrow

lambda abstraction

elimination rules

\longleftrightarrow

function application

assumption rule

\longleftrightarrow

variables

how to read proof terms

$$M := (\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

how to read proof terms

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this proof term M is a function
that takes as its first argument a proof of $a \rightarrow b$ called f
and as its second argument a proof of a called x
and then maps those to a proof of b

how to read proof terms

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the argument f is itself a function that
maps proofs of a to proofs of b
(conform the BHK-interpretation)

x is an inhabitant of the type a
which corresponds to the proposition a

how to read proof terms

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the argument f is itself a function that
maps proofs of a to proofs of b
(conform the BHK-interpretation)

x is an inhabitant of the type a
which corresponds to the proposition a

to get a proof of b , the function f is applied to x
and the result of that application then is the result of M

the example

Coq

the example

```
Parameter a b : Prop.      a is declared  
                           b is declared
```

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 1)

Lemma one :

(a -> b) -> a -> b.

=====

(a -> b) -> a -> b

$(a \rightarrow b) \rightarrow a \rightarrow b$

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 2)

Lemma one :

f : a -> b

(a -> b) -> a -> b.

=====

intro f.

a -> b

$$\frac{a \rightarrow b}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 3)

Lemma one :

f : a -> b

(a -> b) -> a -> b.

x : a

intro f.

=====

intro x.

b

$$\frac{\frac{b}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 4)

Lemma one :

f : a -> b

(a -> b) -> a -> b.

x : a

intro f.

=====

intro x.

a

apply f.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad a}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

Coq

the example

Parameter a b : Prop.

No more subgoals.

Lemma one :

(a -> b) -> a -> b.

intro f.

intro x.

apply f.

exact x.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

the example

Parameter a b : Prop.

Lemma one :

(a -> b) -> a -> b.

intro f.

intro x.

apply f.

exact x.

Qed.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

Coq

the example

Parameter a b : Prop.

one

: (a -> b) -> a -> b

Lemma one :

(a -> b) -> a -> b.

intro f.

intro x.

apply f.

exact x.

Qed.

Check one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

Coq

the example

Parameter a b : Prop.

```
one = fun (f : a -> b) (x : a) => f x
      : (a -> b) -> a -> b
```

Lemma one :

```
(a -> b) -> a -> b.
intro f.
intro x.
apply f.
exact x.
Qed.
```

Arguments one _%function_scope

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$$(\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 3)

Lemma one :

f : a -> b

(a -> b) -> a -> b.

x : a

intro f.

=====

intro x.

b

apply f.

exact x.

Qed.

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$$(\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 3)

Lemma one :

H : a -> b

(a -> b) -> a -> b.

H0 : a

intros.

=====

apply f.

b

exact x.

Qed.

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^H] \quad [a^{H_0}]}{b} E \rightarrow}{a \rightarrow b} I[H_0] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[H] \rightarrow$$

$$(\lambda H : a \rightarrow b. \lambda H_0 : a. H H_0) : (a \rightarrow b) \rightarrow a \rightarrow b$$

Coq

the example

Parameter a b : Prop.

1 subgoal (ID 3)

Lemma one :

f : a -> b

(a -> b) -> a -> b.

x : a

intros f x.

=====

apply f.

b

exact x.

Qed.

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$$(\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

Coq

the example

Parameter a b : Prop. No more subgoals.

Lemma one :

(a -> b) -> a -> b.

intros f x.

apply f.

exact x.

Qed.

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$$(\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

Coq

the example

Parameter a b : Prop.

No more subgoals.

Lemma one :

(a -> b) -> a -> b.

intros f x.

apply f.

apply x.

Qed.

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$$(\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$$

Coq

the example

Parameter a b : Prop. No more subgoals.

Lemma one :

(a -> b) -> a -> b.

auto.

Qed.

Check one.

Print one.

Coq

the example

Parameter a b : Prop.

```
one = fun H : a -> b => H
      : (a -> b) -> a -> b
```

Lemma one :

```
(a -> b) -> a -> b.
```

```
Arguments one _%function_scope
```

auto.

Qed.

Check one.

Print one.

$$\frac{[a \rightarrow b^H]}{(a \rightarrow b) \rightarrow a \rightarrow b} I[H] \rightarrow$$

```
(λH : a → b. H) : (a → b) → a → b
```

Coq

the example

Parameter a b : Prop.

```
one = fun (f : a -> b) (x : a) => f x
      : (a -> b) -> a -> b
```

Lemma one :

```
(a -> b) -> a -> b.
```

```
intros f x.
```

```
apply f.
```

```
apply x.
```

```
Qed.
```

Check one.

Print one.

$$\frac{\frac{\frac{[a \rightarrow b^f] \quad [a^x]}{b} E \rightarrow}{a \rightarrow b} I[x] \rightarrow}{(a \rightarrow b) \rightarrow a \rightarrow b} I[f] \rightarrow$$

$(\lambda f : a \rightarrow b. \lambda x : a. fx) : (a \rightarrow b) \rightarrow a \rightarrow b$

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 2)

Lemma two :

a \vee b \rightarrow b \vee a.

=====

a \vee b \rightarrow b \vee a

$$a \vee b \rightarrow b \vee a$$

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 3)

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

x : a \vee b

=====

b \vee a

$$\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow$$

example with disjunction

Parameter a b : Prop.

2 subgoals (ID 4)

Lemma two :

x : a \vee b

a \vee b \rightarrow b \vee a.

=====

intro x.

a \rightarrow b \vee a

elim x.

subgoal 2 (ID 5) is:

b \rightarrow b \vee a

$$\frac{[a \vee b^x] \quad a \rightarrow b \vee a \quad b \rightarrow b \vee a}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} EV$$

example with disjunction

Parameter a b : Prop.

2 subgoals (ID 6)

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

x : a \vee b

y : a

=====

b \vee a

subgoal 2 (ID 5) is:

b \rightarrow b \vee a

$$\frac{[a \vee b^x] \quad \frac{b \vee a}{a \rightarrow b \vee a} I[y] \rightarrow \quad b \rightarrow b \vee a}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} E\vee$$

example with disjunction

Parameter a b : Prop.

2 subgoals (ID 8)

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

right.

x : a \vee b

y : a

=====

a

subgoal 2 (ID 5) is:

b \rightarrow b \vee a

$$\frac{\frac{a}{b \vee a} \text{Ir}\vee \quad \frac{a \rightarrow b \vee a}{a \rightarrow b \vee a} I[y] \rightarrow}{[a \vee b^x] \quad \frac{b \rightarrow b \vee a}{a \vee b \rightarrow b \vee a} E\vee} \frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow$$

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 5)

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

right.

apply y.

x : a \vee b

=====

b \rightarrow b \vee a

$$\frac{\frac{[a \vee b^x] \quad \frac{\frac{[a^y] \quad Ir\vee}{b \vee a}}{a \rightarrow b \vee a} I[y] \rightarrow}{b \vee a} E\vee}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} E\vee$$

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 9)

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

right.

apply y.

intro z.

x : a \vee b

z : b

=====

b \vee a

$$\frac{\frac{[a \vee b^x]}{a \rightarrow b \vee a} I[y] \rightarrow \quad \frac{\frac{[a^y]}{b \vee a} Ir \vee \quad b \vee a}{b \rightarrow b \vee a} I[z] \rightarrow}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} EV$$

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 11)

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

right.

apply y.

intro z.

left.

x : a \vee b

z : b

=====

b

$$\frac{\frac{[a \vee b^x]}{a \rightarrow b \vee a} I[\vee] \rightarrow \quad \frac{\frac{[a^y]}{b \vee a} Ir\vee \quad \frac{b}{b \vee a} Il\vee}{b \rightarrow b \vee a} I[\rightarrow] \rightarrow}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} EV$$

example with disjunction

Parameter a b : Prop.

No more subgoals.

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

right.

apply y.

intro z.

left.

apply z.

$$\frac{[a \vee b^x] \quad \frac{\frac{[a^y]}{b \vee a} Ir\vee \quad \frac{[b^z]}{b \vee a} Il\vee}{a \rightarrow b \vee a} I[y] \rightarrow \quad \frac{b \vee a \quad I[z] \rightarrow}{b \rightarrow b \vee a} EV}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} EV$$

example with disjunction

Parameter a b : Prop.

Lemma two :

a \vee b \rightarrow b \vee a.

intro x.

elim x.

intro y.

right.

apply y.

intro z.

left.

apply z.

Qed.

$$\frac{[a \vee b^x] \quad \frac{\frac{[a^y]}{b \vee a} Ir\vee \quad \frac{[b^z]}{b \vee a} Il\vee}{a \rightarrow b \vee a} I[y] \rightarrow \quad \frac{b \rightarrow b \vee a}{I[z] \rightarrow} EV}{\frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow} EV$$

example with disjunction

Parameter a b : Prop.

2 subgoals (ID 4)

Lemma two :

x : a \vee b

a \vee b \rightarrow b \vee a.

=====

intro x.

a \rightarrow b \vee a

elim x.

intro y.

subgoal 2 (ID 5) is:

right.

b \rightarrow b \vee a

apply y.

intro z.

left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

2 subgoals (ID 4)

Lemma two :

x : a \vee b

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intro x.

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elim x.

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subgoal 2 (ID 5) is:

right.

b \rightarrow b \vee a

apply y.

- intro z.

left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 4)

Lemma two :

x : a \vee b

 a \vee b \rightarrow b \vee a.

=====

intro x.

a \rightarrow b \vee a

elim x.

- intro y.

 right.

 apply y.

- intro z.

 left.

 apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 6)

Lemma two :

a \vee b \rightarrow b \vee a.

x : a \vee b

y : a

=====

b \vee a

intro x.

elim x.

- intro y.

right.

apply y.

- intro z.

left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 8)

Lemma two :

a \vee b \rightarrow b \vee a.

x : a \vee b

y : a

=====

intro x.

elim x.

a

- intro y.

right.

apply y.

- intro z.

left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 5)

Lemma two :

subgoal 1 (ID 5) is:

a \vee b \rightarrow b \vee a.

b \rightarrow b \vee a

intro x.

elim x.

- intro y.

right.

apply y.

- intro z.

left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 5)

Lemma two :

x : a \vee b

 a \vee b \rightarrow b \vee a.

=====

intro x.

b \rightarrow b \vee a

elim x.

- intro y.

 right.

 apply y.

- intro z.

 left.

 apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 6)

Lemma two :

a \vee b \rightarrow b \vee a.

x : a \vee b

y : a

intro x.

=====

elim x.

b \vee a

- intro y.

right.

apply y.

- intro z.

left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 9)

Lemma two :

y : a

a \vee b \rightarrow b \vee a.

=====

intro x.

b \vee a

destruct x as [y|z].

- right.

 apply y.

- left.

 apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 9)

Lemma two :

a \vee b \rightarrow b \vee a.

y : a

=====

intro x.

b \vee a

destruct x as [y|z].

- right.

 apply y.

- left.

 apply z.

Qed.

example with disjunction

Parameter a b : Prop.

1 subgoal (ID 8)

Lemma two :

a \vee b \rightarrow b \vee a.

y : a

=====

intros [y|z].

b \vee a

- right.

apply y.

- left.

apply z.

Qed.

example with disjunction

Parameter a b : Prop.

Lemma two :

```
a ∨ b → b ∨ a.  
intros [y|z].  
- right.  
  apply y.  
- left.  
  apply z.  
Qed.
```

$$\frac{\frac{[a^y]}{b \vee a} Ir\vee \quad \frac{[b^z]}{b \vee a} Il\vee}{\frac{[a \vee b^x]}{a \rightarrow b \vee a} I[y] \rightarrow \quad \frac{[a \vee b^x]}{b \rightarrow b \vee a} I[z] \rightarrow} EV \quad \frac{b \vee a}{a \vee b \rightarrow b \vee a} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop.

1 subgoal (ID 3)

Lemma three :

a /\ b -> b /\ a.

=====

a /\ b -> b /\ a

$$a \wedge b \rightarrow b \wedge a$$

example with conjunction

Parameter a b : Prop.

1 subgoal (ID 8)

Lemma three :

a /\ b -> b /\ a.

intros [y z].

y : a

z : b

=====

b /\ a

$$\frac{\frac{\frac{b \wedge a}{b \rightarrow b \wedge a} I[z] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{b \wedge a} E \wedge}{a \wedge b \rightarrow b \wedge a} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop.

2 subgoals (ID 10)

Lemma three :

a /\ b -> b /\ a.

intros [y z].

split.

y : a

z : b

=====

b

subgoal 2 (ID 11) is:

a

$$\frac{\frac{\frac{b \quad a}{b \wedge a} I\wedge \quad \frac{b \wedge a}{b \rightarrow b \wedge a} I[z] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{[a \wedge b^x] \quad \frac{\quad}{b \wedge a} E\wedge} E\wedge \quad \frac{b \wedge a}{a \wedge b \rightarrow b \wedge a} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop.

1 subgoal (ID 10)

Lemma three :

a /\ b -> b /\ a.

intros [y z].

split.

-

y : a

z : b

=====

b

$$\frac{\frac{\frac{b}{b \wedge a} I \wedge \quad \frac{a}{b \rightarrow b \wedge a} I[z] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{[a \wedge b^x] \quad \frac{a \rightarrow b \rightarrow b \wedge a}{E \wedge}} E \wedge \quad \frac{b \wedge a}{a \wedge b \rightarrow b \wedge a} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop.

1 subgoal (ID 11)

Lemma three :

subgoal 1 (ID 11) is:

a /\ b -> b /\ a.

a

intros [y z].

split.

- apply z.

$$\frac{\frac{\frac{[b^z] \quad a}{b \wedge a} I \wedge}{b \rightarrow b \wedge a} I[z] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{[a \wedge b^x] \quad \frac{\quad}{b \wedge a} E \wedge} \frac{\quad}{a \wedge b \rightarrow b \wedge a} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop.

1 subgoal (ID 11)

Lemma three :

a /\ b -> b /\ a.

intros [y z].

split.

- apply z.

-

y : a

z : b

=====

a

$$\frac{\frac{\frac{[b^z] \quad a}{b \wedge a} I\wedge \quad \frac{b \wedge a}{b \rightarrow b \wedge a} I[z] \rightarrow \quad \frac{I[y] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} E\wedge}{[a \wedge b^x] \quad \frac{a \rightarrow b \rightarrow b \wedge a}{b \wedge a} E\wedge}{a \wedge b \rightarrow b \wedge a} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop. No more subgoals.

Lemma three :

 a /\ b -> b /\ a.

intros [y z].

split.

- apply z.

- apply y.

$$\frac{\frac{\frac{[b^z] \quad [a^y]}{b \wedge a} I \wedge}{b \rightarrow b \wedge a} I[z] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{[a \wedge b^x] \quad \frac{b \wedge a}{a \wedge b \rightarrow b \wedge a} E \wedge} I[x] \rightarrow$$

example with conjunction

Parameter a b : Prop.

Lemma three :

a /\ b -> b /\ a.

intros [y z].

split.

- apply z.

- apply y.

Qed.

$$\frac{\frac{\frac{[b^z] \quad [a^y]}{b \wedge a} I \wedge}{b \rightarrow b \wedge a} I[z] \rightarrow}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{[a \wedge b^x] \quad \frac{b \wedge a}{a \wedge b \rightarrow b \wedge a} I[x] \rightarrow} E \wedge$$

example with conjunction

Parameter a b : Prop.

Lemma three :

```
a /\ b -> b /\ a.  
intros [y z].  
split.  
- apply z.  
- apply y.  
Qed.
```

$$\frac{\frac{\frac{[b^z] \quad [a^y]}{b \wedge a} I\wedge \quad \frac{I[z] \rightarrow}{b \rightarrow b \wedge a}}{a \rightarrow b \rightarrow b \wedge a} I[y] \rightarrow}{E\wedge} \frac{[a \wedge b^x]}{b \wedge a} I[x] \rightarrow}{a \wedge b \rightarrow b \wedge a}$$

tactics for proof rules

| | |
|---------------------------|--------------------------------------|
| $I \rightarrow I \neg$ | intro intros |
| $E \rightarrow E \neg$ | apply |
| ass | exact apply |
| $E \wedge E \vee E \perp$ | elim destruct intros with pattern |
| $I \wedge$ | split |
| $I l \vee$ | left |
| $I r \vee$ | right |
| $I \top$ | apply I |

tactics for proof rules

| | | |
|---------------------------|--------------------------------------|----------|
| $I \rightarrow I \neg$ | intro intros | |
| $E \rightarrow E \neg$ | apply | |
| ass | exact apply | |
| $E \wedge E \vee E \perp$ | elim destruct intros with pattern | |
| $I \wedge$ | split | |
| $I l \vee$ | left | |
| $I r \vee$ | right | |
| $I \top$ | apply I | I : True |

bullets

structure tactic scripts according to subgoals

related bullets need to match:

- -- --- ---- etc.

+ ++ +++ ++++ etc.

* ** *** **** etc.

intro patterns

only works with `intros`, not with `intro`

the pattern needs to match the shape of assumptions

| goal | tactic |
|---|--------------------------------------|
| $A \rightarrow G$ | <code>intros HA.</code> |
| $A \rightarrow B \rightarrow G$ | <code>intros HA HB.</code> |
| $A \wedge B \rightarrow G$ | <code>intros [HA HB].</code> |
| $A \vee B \rightarrow G$ | <code>intros [HA HB].</code> |
| $A \vee (B \wedge C) \rightarrow D \rightarrow G$ | <code>intros [HA [HB HC]] HD.</code> |

apply

$$H : A_1 \rightarrow \cdots \rightarrow A_n \rightarrow B$$

goal B $\xrightarrow{\text{apply } H}$ new goals A_1, \dots, A_n

apply

the number n of antecedents of H may be zero
 H not only a variable, may be an arbitrary term

$$H : A_1 \rightarrow \dots \rightarrow A_n \rightarrow B$$

goal B $\xrightarrow{\text{apply } H}$ new goals A_1, \dots, A_n

apply

the number n of antecedents of H may be zero
 H not only a variable, may be an arbitrary term

$$H : A_1 \rightarrow \cdots \rightarrow A_n \rightarrow B$$

goal B $\xrightarrow{\text{apply } H}$ new goals A_1, \dots, A_n

$$\frac{[A_1 \rightarrow \cdots \rightarrow A_n \rightarrow B^H]}{A_2 \rightarrow \cdots \rightarrow A_n \rightarrow B} \quad A_1 \quad E \rightarrow$$
$$\frac{\quad}{A_2 \rightarrow \cdots \rightarrow A_n \rightarrow B} \quad A_2 \quad E \rightarrow$$

⋮

$$\frac{A_{n-1} \rightarrow A_n \rightarrow B}{A_n \rightarrow B} \quad A_{n-1} \quad E \rightarrow$$
$$\frac{\quad}{A_n \rightarrow B} \quad A_n \quad E \rightarrow$$
$$\frac{\quad}{B}$$

elim

$$H : A_1 \rightarrow \dots \rightarrow A_n \rightarrow B \otimes C$$

`elim` and `destruct` eliminate the connective \otimes
which leaves the goal unchanged

but also generate extra subgoals A_1, \dots, A_n

summary

| minimal logic | type theory | Coq |
|--------------------|----------------------|-------|
| introduction rules | lambda abstraction | intro |
| elimination rules | function application | apply |

summary

| logic | type theory | Coq |
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| introduction rules | lambda abstraction | <code>intro</code> |
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| introduction rules | (in two weeks) | (various) |
| elimination rules | (in two weeks) | <code>elim</code> |

summary

| logic | type theory | Coq |
|--------------------|----------------------|--------------------|
| introduction rules | lambda abstraction | <code>intro</code> |
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for more about the tactics
read the Coq manual!



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