propositional logic & simple types

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Type Theory & Coq
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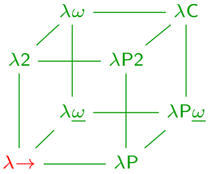


type systems and logics

teaser: a proof term for a proof

$$\lambda x : a \to b \to c.\, \lambda y : a \to b.\, \lambda z : a.\, xz(yz)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \quad [a^z]}{b \to c}\, E{\to} \qquad \dfrac{[a \to b^y] \quad [a^z]}{b}\, E{\to}}{c}\, E{\to}}{a \to c}\, I[z]{\to}}{(a \to b) \to a \to c}\, I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\, I[x]{\to}$$

many systems

- one untyped lambda calculus

  (last week: recap)

- many typed lambda calculi = type theories

  - **Church-style**
    variables explicitly typed

    the systems in this course
    Coq!

  - **Curry-style**
    assigning types to untyped terms

    (in one of Herman's lectures)

- many logics

Curry-Howard correspondence

'propositions-as-types'

type systems	$\sim$	logics
datatypes	$\sim$	propositions
$A \times B$	$\sim$	$A \wedge B$
$A \to B$	$\sim$	$A \to B$
objects of a datatype	$\sim$	proofs of a proposition
'inhabitants'		'proof objects'
non-empty type	$\sim$	true proposition
empty type	$\sim$	false proposition

Curry-Howard correspondence

'propositions-as-types'

type systems	$\sim$	logics
datatypes	$\sim$	propositions
$A \times B$	$\sim$	$A \wedge B$
$A \to B$	$\sim$	$A \to B$
objects of a datatype	$\sim$	proofs of a proposition
'inhabitants'		'proof objects'
non-empty type	$\sim$	true proposition
empty type	$\sim$	false proposition
derivation in a type theory	$\sim$	derivation in a logic

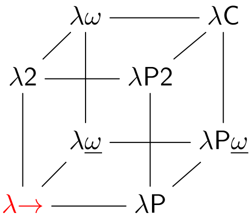
type systems in this course

logics	$\sim$	type systems		
propositional logic	$\sim$	$\lambda{\to}$	$=$	simply typed lambda calculus
predicate logic	$\sim$	$\lambda\mathrm{P}$	$=$	dependently typed lambda calculus
second order logic	$\sim$	$\lambda 2$	$=$	polymorphic lambda calculus
higher order logic	$\sim$	$\lambda\mathrm{C}$	$=$	CC $=$ Calculus of Constructions
'the logic of Coq'	$\sim$	CIC	$=$	Calculus of Inductive Constructions

$$\begin{tikzcd}
& \lambda\omega \arrow[rr, dash] & & \lambda\mathrm{C} \\
\lambda 2 \arrow[ur, dash] \arrow[rr, dash] & & \lambda\mathrm{P}2 \arrow[ur, dash] & \\
& \lambda\underline{\omega} \arrow[uu, dash] \arrow[rr, dash] & & \lambda\mathrm{P}\underline{\omega} \arrow[uu, dash] \\
\lambda{\to} \arrow[uu, dash] \arrow[ur, dash] \arrow[rr, dash] & & \lambda\mathrm{P} \arrow[uu, dash] \arrow[ur, dash] &
\end{tikzcd}$$

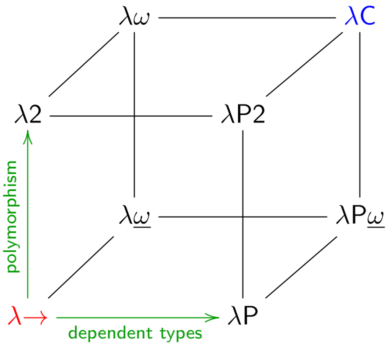
type systems in this course

logics	$\sim$	type systems		
propositional logic	$\sim$	$\lambda{\to}$	$=$	simply typed lambda calculus
predicate logic	$\sim$	λP	$=$	dependently typed lambda calculus
second order logic	$\sim$	$\lambda 2$	$=$	polymorphic lambda calculus
higher order logic	$\sim$	λC	$=$	CC $=$ Calculus of Constructions
'the logic of Coq'	$\sim$	CIC	$=$	Calculus of Inductive Constructions



the Barendregt cube
eight PTSs = pure type systems

(not explicitly in Femke's course notes)



- **natural deduction** $B_1, \ldots, B_m \vdash A$

  introduction and elimination rules for each connective

$$\frac{\ldots \vdash \ldots}{\ldots \vdash \cdots \otimes \cdots}\, I \qquad\qquad \frac{\ldots \vdash \cdots \otimes \cdots}{\ldots \vdash \ldots}\, E$$

  - Gentzen–Prawitz style: proof is a derivation tree
  - Jaśkowski–Fitch style: proof consists of nested boxes / flags

- **sequent calculus**: LK & LJ $B_1, \ldots, B_m \vdash A_1, \ldots, A_n$

  left and right rules for each connective

$$\frac{\ldots \vdash \ldots}{\cdots \otimes \cdots \vdash \ldots}\, L \qquad\qquad \frac{\ldots \vdash \ldots}{\ldots \vdash \cdots \otimes \cdots}\, R$$

- **Hilbert-style** $\vdash A$

  axioms for each connective (and at most two rules)

- **natural deduction** $B_1, \ldots, B_m \vdash A$

  introduction and elimination rules for each connective

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- **sequent calculus**: LK & LJ $B_1, \ldots, B_m \vdash A_1, \ldots, A_n$

  left and right rules for each connective

  $$\frac{\ldots \vdash \ldots}{\cdots \otimes \cdots \vdash \ldots}\, L \qquad\qquad \frac{\ldots \vdash \ldots}{\ldots \vdash \cdots \otimes \cdots}\, R$$

- **Hilbert-style** $\vdash A$

  axioms for each connective (and at most two rules)

defining a logic or type system

- **syntax**
  - terms
  - types
  - formulas
  - contexts
  - judgments
- **rules**
  - logics: proof rules
  - type systems: typing rules
- **reduction**

defining a logic or type system

- **syntax**
  - terms
  - types
  - formulas
  - contexts
  - judgments

$$M, N ::= x \mid M\,N \mid \lambda x.\, M$$

- **rules**
  - logics: proof rules
  - type systems: typing rules

no rules for untyped lambda calculus

- **reduction**

$$(\lambda x.\, M)\, N \to_\beta M[x := N]$$

three propositional logics

- **minimal propositional logic**
  the logic that corresponds to simply typed lambda calculus
  only implication: $\to$
- **constructive propostional logic**
  all connectives: $\to$, $\wedge$, $\vee$, $\neg$, $\bot$, $\top$
- **classical propositional logic**

$$A \vee \neg A$$
$$\neg\neg A \to A$$

propositional logic: syntax

formulas

$$A, B ::= a \mid A \to B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \bot$$

a, b, c, ...	atomic propositions later: propositional variables
A, B, C, ...	meta-variables for arbitrary formulas

implication associates to the right:

$$a \to b \to c \quad \text{means} \quad a \to (b \to c)$$

order of binding strength: $\neg > \wedge > \vee > \to$

$$a \vee \neg b \wedge c \to d \quad \text{means} \quad (a \vee ((\neg b) \wedge c)) \to d$$

the two rules of minimal propositional logic

the introduction and elimination rules of implication:

$$\dfrac{\begin{array}{c}[A^x]\\ \vdots\\ B\end{array}}{A \to B}\, I[x]{\to} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \to B\end{array} \qquad \begin{array}{c}\vdots\\ A\end{array}}{B}\, E{\to}$$

in the introduction rule the assumption $[A^x]$ may be used an arbitrary number of times: zero, one or more

in the elimination rule the proof of $A \to B$ is on the *left* of the proof of A

example proof of minimal propositional logic

$$(a \to b \to c) \to (a \to b) \to a \to c$$

example proof of minimal propositional logic

$[a \to b \to c^x]$

$$\frac{(a \to b) \to a \to c}{(a \to b \to c) \to (a \to b) \to a \to c}\, I[x]{\to}$$

example proof of minimal propositional logic

$[a \to b \to c^x]$ $\quad [a \to b^y]$

$$\dfrac{\dfrac{a \to c}{(a \to b) \to a \to c}\, I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\, I[x]{\to}$$

example proof of minimal propositional logic

$[a \to b \to c^x] \quad [a \to b^y] \quad [a^z]$

$$\dfrac{\dfrac{\dfrac{c}{a \to c}\, I[z]{\to}}{(a \to b) \to a \to c}\, I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\, I[x]{\to}$$

example proof of minimal propositional logic

$[a \to b \to c^x] \quad [a \to b^y] \quad [a^z]$

$$\dfrac{\dfrac{\dfrac{\dfrac{b \to c \qquad\qquad\qquad b}{c}\,E{\to}}{a \to c}\,I[z]{\to}}{(a \to b) \to a \to c}\,I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\,I[x]{\to}$$

example proof of minimal propositional logic

$[a \to b \to c^x] \quad [a \to b^y] \quad [a^z]$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a \to b \to c \qquad a}{b \to c}\,E\!\to \qquad b}{c}\,E\!\to}{a \to c}\,I[z]\!\to}{(a \to b) \to a \to c}\,I[y]\!\to}{(a \to b \to c) \to (a \to b) \to a \to c}\,I[x]\!\to$$

example proof of minimal propositional logic

$[a \to b \to c^x]$ $\quad$ $[a \to b^y]$ $\quad$ $[a^z]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \qquad a}{b \to c}\,E{\to} \qquad b}{c}\,E{\to}}{a \to c}\,I[z]{\to}}{(a \to b) \to a \to c}\,I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\,I[x]{\to}$$

example proof of minimal propositional logic

$[a \to b \to c^x]$ $[a \to b^y]$ $[a^z]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \quad [a^z]}{b \to c}\,E{\to} \qquad b}{c}\,E{\to}}{a \to c}\,I[z]{\to}}{(a \to b) \to a \to c}\,I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\,I[x]{\to}$$

example proof of minimal propositional logic

$[a \to b \to c^x] \quad [a \to b^y] \quad [a^z]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \qquad [a^z]}{b \to c}\,E{\to} \qquad \dfrac{a \to b \qquad a}{b}\,E{\to}}{c}\,E{\to}}{\dfrac{a \to c}{}}\,I[z]{\to}}{}
$$

$$
\cfrac{\cfrac{\cfrac{c}{a \to c}\,I[z]{\to}}{(a \to b) \to a \to c}\,I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\,I[x]{\to}
$$

example proof of minimal propositional logic

$[a \to b \to c^x] \quad [a \to b^y] \quad [a^z]$

$$
\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \qquad [a^z]}{b \to c}\, E{\to} \qquad \dfrac{[a \to b^y] \qquad a}{b}\, E{\to}}{\dfrac{\dfrac{c}{a \to c}\, I[z]{\to}}{(a \to b) \to a \to c}\, I[y]{\to}}\, E{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\, I[x]{\to}
$$

example proof of minimal propositional logic

$[a \to b \to c^x] \quad [a \to b^y] \quad [a^z]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a \to b \to c^x] \quad [a^z]}{b \to c}\,E{\to} \qquad \dfrac{[a \to b^y] \quad [a^z]}{b}\,E{\to}}{c}\,E{\to}}{a \to c}\,I[z]{\to}}{(a \to b) \to a \to c}\,I[y]{\to}}{(a \to b \to c) \to (a \to b) \to a \to c}\,I[x]{\to}
$$

example proof of minimal propositional logic

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[a \to b \to c^x] \quad [a^z]}{b \to c}\,E{\to} \qquad \cfrac{[a \to b^y] \quad [a^z]}{b}\,E{\to}}{c}\,E{\to}}{a \to c}\,I[z]{\to}}{(a \to b) \to a \to c}\,I[y]{\to}}{\color{red}{(a \to b \to c) \to (a \to b) \to a \to c}}\,I[x]{\to}
$$

constructive propositional logic: the other rules

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array} \qquad \begin{array}{c}\vdots\\ B\end{array}}{A \wedge B}\;I\wedge \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{A}\;El\wedge \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{B}\;Er\wedge$$

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B}\;Il\vee \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B}\;Ir\vee \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array} \qquad \begin{array}{c}\vdots\\ A \to C\end{array} \qquad \begin{array}{c}\vdots\\ B \to C\end{array}}{C}\;E\vee$$

$$\dfrac{\begin{array}{c}[A^x]\\ \vdots\\ \bot\end{array}}{\neg A}\;I\neg \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \neg A\end{array} \qquad \begin{array}{c}\vdots\\ A\end{array}}{\bot}\;E\neg \qquad\qquad \dfrac{}{\top}\;I\top \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{A}\;E\bot$$

constructive propositional logic: the other rules

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ A & & B\end{matrix}}{A \wedge B}\, I\wedge \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\, El\wedge \qquad \dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{B}\, Er\wedge$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\, Il\vee \qquad \dfrac{\begin{matrix}\vdots \\ B\end{matrix}}{A \vee B}\, Ir\vee \qquad \dfrac{\begin{matrix}\vdots & & \vdots & & \vdots \\ A \vee B & & A \to C & & B \to C\end{matrix}}{C}\, E\vee$$

$$\dfrac{\begin{matrix}[A^x] \\ \vdots \\ \bot\end{matrix}}{\neg A}\, I\neg \qquad \dfrac{\begin{matrix}\vdots & & \vdots \\ \neg A & & A\end{matrix}}{\bot}\, E\neg \qquad \dfrac{}{\top}\, I\top \qquad \dfrac{\begin{matrix}\vdots \\ \bot\end{matrix}}{A}\, E\bot$$

$$\neg A := A \to \bot$$

variants of elimination rules

- what Coq's `elim` tactic implements for conjunction:

$$\dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array} \qquad \begin{array}{c}\vdots\\ A \to B \to C\end{array}}{C}\; E\wedge$$

- often the disjunction elimination rule is:

$$\dfrac{\begin{array}{c}\\ \vdots\\ A \vee B\end{array} \qquad \begin{array}{c}[A^x]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B^y]\\ \vdots\\ C\end{array}}{C}\; E[x,y]\vee$$

not what Coq's `elim` tactic implements for disjunction

example proof beyond minimal propositional logic

$$a \vee b \to b \vee a$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$$\dfrac{b \vee a}{a \vee b \to b \vee a}\; I[x]{\to}$$

example proof beyond minimal propositional logic

$[a \vee b^x]$

$$\cfrac{\cfrac{[a \vee b^x] \qquad a \to b \vee a \qquad b \to b \vee a}{b \vee a}\, E\vee}{a \vee b \to b \vee a}\, I[x]{\to}$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $\quad$ $[a^y]$

$$\dfrac{[a \vee b^x] \qquad \dfrac{b \vee a}{a \to b \vee a}\,I[y]{\to} \qquad b \to b \vee a}{\dfrac{b \vee a}{a \vee b \to b \vee a}\,I[x]{\to}}\,E\vee$$

example proof beyond minimal propositional logic

$[a \vee b^x] \quad [a^y]$

$$\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{a}{b \vee a}\,Ir{\vee}}{a \to b \vee a}\,I[y]{\to} \qquad b \to b \vee a}{\dfrac{b \vee a}{a \vee b \to b \vee a}\,I[x]{\to}}\,E{\vee}$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $[a^y]$

$$
\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\, Ir\vee}{a \to b \vee a}\, I[y]{\to} \qquad b \to b \vee a}{\dfrac{b \vee a}{a \vee b \to b \vee a}\, I[x]{\to}}\, E\vee
$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $\qquad$ $[b^z]$

$$
\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{\dfrac{[a^y]}{b \vee a}\, Ir\vee}{a \to b \vee a}\, I[y]{\to} \qquad \dfrac{b \vee a}{b \to b \vee a}\, I[z]{\to}}{\dfrac{b \vee a}{a \vee b \to b \vee a}\, I[x]{\to}}\, E\vee
$$

example proof beyond minimal propositional logic

$[a \vee b^x]$ $\qquad$ $[b^z]$

$$
\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\,Ir\vee}{a \to b \vee a}\,I[y]{\to} \qquad \dfrac{\dfrac{b}{b \vee a}\,Il\vee}{b \to b \vee a}\,I[z]{\to}}{\dfrac{b \vee a}{a \vee b \to b \vee a}\,I[x]{\to}}\,E\vee
$$

$[a \vee b^x]$ $\qquad$ $[b^z]$

$$
\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\, Ir\vee}{a \to b \vee a}\, I[y]{\to} \qquad \dfrac{\dfrac{[b^z]}{b \vee a}\, Il\vee}{b \to b \vee a}\, I[z]{\to}}{\dfrac{b \vee a}{a \vee b \to b \vee a}\, I[x]{\to}}\, E\vee
$$

example proof beyond minimal propositional logic

$$
\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\, Ir\vee}{a \to b \vee a}\, I[y]{\to} \qquad \dfrac{\dfrac{[b^z]}{b \vee a}\, Il\vee}{b \to b \vee a}\, I[z]{\to}}{\dfrac{b \vee a}{\color{red}{a \vee b \to b \vee a}}\, I[x]{\to}}\, E\vee
$$

alternative style: explicit assumption lists

syntax

$$
\begin{aligned}
A, B &::= a \mid A \to B \mid \ldots && \text{formulas} \\
\Gamma &::= \cdot \mid \Gamma, A && \text{assumption lists} \\
\mathcal{J} &::= \Gamma \vdash A && \text{sequents}
\end{aligned}
$$

we do not write the dot and the comma after the dot
still natural deduction, *not* sequent calculus

rules

$$\frac{}{\Gamma \vdash A}\,\text{ass} \qquad \text{for } A \in \Gamma$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\, I{\to} \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\, E{\to} \qquad \ldots$$

the example proofs with explicit assumption lists

$\Gamma_1 := a \to b \to c,\ a \to b,\ a$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\Gamma_1 \vdash a \to b \to c}\,\text{ass}
          \quad
          \dfrac{}{\Gamma_1 \vdash a}\,\text{ass}
        }{\Gamma_1 \vdash b \to c}\,E{\to}
        \quad
        \dfrac{
          \dfrac{}{\Gamma_1 \vdash a \to b}\,\text{ass}
          \quad
          \dfrac{}{\Gamma_1 \vdash a}\,\text{ass}
        }{\Gamma_1 \vdash b}\,E{\to}
      }{\Gamma_1 \vdash c}\,E{\to}
    }{a \to b \to c,\ a \to b \vdash a \to c}\,I{\to}
  }{a \to b \to c \vdash (a \to b) \to a \to c}\,I{\to}
}{\vdash (a \to b \to c) \to (a \to b) \to a \to c}\,I{\to}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{a \vee b \vdash a \vee b}\,\text{ass}
    \quad
    \dfrac{
      \dfrac{
        \dfrac{}{a \vee b,\ a \vdash a}\,\text{ass}
      }{a \vee b,\ a \vdash b \vee a}\,Ir{\vee}
    }{a \vee b \vdash a \to b \vee a}\,I{\to}
    \quad
    \dfrac{
      \dfrac{
        \dfrac{}{a \vee b,\ b \vdash b}\,\text{ass}
      }{a \vee b,\ b \vdash b \vee a}\,Il{\vee}
    }{a \vee b \vdash b \to b \vee a}\,I{\to}
  }{a \vee b \vdash b \vee a}\,E{\vee}
}{\vdash a \vee b \to b \vee a}\,I{\to}
$$

the S combinator

$$\lambda xyz.\, xz(yz)$$

the S combinator

$$\lambda xyz.\, xz(yz)$$

$$(\lambda x.\, (\lambda y.\, (\lambda z.\, ((xz)(yz)))))$$

the S combinator

untyped
typed: Curry-style

$$\lambda xyz.\, xz(yz)$$

$$(\lambda x.\, (\lambda y.\, (\lambda z.\, ((xz)(yz)))))$$

typed: Church-style

$$\lambda x : a \to b \to c.\, \lambda y : a \to b.\, \lambda z : a.\, xz(yz)$$

the S combinator

untyped
typed: Curry-style

$$\lambda xyz.\, xz(yz)$$

$$(\lambda x.\, (\lambda y.\, (\lambda z.\, ((xz)(yz)))))$$

typed: Church-style

$$\lambda x : a \to b \to c.\, \lambda y : a \to b.\, \lambda z : a.\, xz(yz)$$

Coq syntax

```
fun x : a -> b -> c => fun y : a -> b => fun z : a =>
  x z (y z)
```

the S combinator

untyped
typed: Curry-style

$$\lambda xyz.\, xz(yz)$$
$$(\lambda x.\, (\lambda y.\, (\lambda z.\, ((xz)(yz)))))$$

typed: Church-style

$$\lambda x : a \to b \to c.\, \lambda y : a \to b.\, \lambda z : a.\, xz(yz)$$

Coq syntax

```
fun x : a -> b -> c => fun y : a -> b => fun z : a =>
  x z (y z)
```

```
fun (x : a -> b -> c) (y : a -> b) (z : a) => x z (y z)
```

BHK interpretation

Brouwer–Heyting–Kolmogorov
$\sim$ Curry-Howard correspondence
constructive 'meaning' of the logical connectives

connection between proofs and lambda calculus

proof of $A \to B$	$=$	function from proofs of A to proofs of B
proof of $A \wedge B$	$=$	pair of a proof of A and a proof of B
proof of $A \vee B$	$=$	either a proof of A, or a proof of B
proof of $\neg A$	$=$	proof of $A \to \bot$
proof of $\top$		the object I
proof of $\bot$		*does not exist*

Brouwer–Heyting–Kolmogorov
$\sim$ Curry-Howard correspondence
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proof of $\top$		the object I
proof of $\bot$		*does not exist*

proof of $a \wedge b \to b \wedge a$ is

a function that inputs a pair $\langle x, y \rangle$, and returns $\langle y, x \rangle$
with x a proof of a and y a proof of b

$$\lambda p : a \wedge b.\, \langle \pi_2 p, \pi_1 p \rangle$$

different names for the same type theory

simply typed lambda calculus

$\parallel$

simple type theory = STT

$\parallel$

$\lambda\!\rightarrow$

3 typing rules

$\nparallel$

7 typing rules

same set of well-typed terms

simply typed lambda calculus: syntax and rules

syntax

$A, B ::= a \mid A \to B$	types
$M, N ::= x \mid MN \mid \lambda x : A.\, M$	preterms
$\Gamma ::= \cdot \mid \Gamma, x : A$	contexts
$\mathcal{J} ::= \Gamma \vdash M : A$	judgments

rules

$$\frac{}{\Gamma \vdash x : A} \quad \text{for } (x : A) \in \Gamma$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

well-typedness

$$\Gamma \vdash M : A \text{ is derivable} \implies M \text{ is well-typed}$$

well-typed preterms are called terms

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

$$\Gamma \vdash M : A$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\underbrace{\cdot}_{\Gamma} \vdash \underbrace{(\lambda f : a \to b.\, \lambda x : a.\, f x)}_{M} : \underbrace{(a \to b) \to a \to b}_{A}$$

$$\Gamma \vdash M : A$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\cdot \vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\cdot \vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\cdot \vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\cdot \vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\underbrace{\cdot}_{\Gamma} \vdash (\lambda \underbrace{f}_{x} : \underbrace{a \to b}_{A}.\, \underbrace{\lambda x : a.\, f x}_{M}) : \underbrace{(a \to b)}_{A} \to \underbrace{a \to b}_{B}$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\frac{\overbrace{\cdot}^{\Gamma},\ \overbrace{f}^{x} : \overbrace{a \to b}^{A} \vdash \overbrace{(\lambda x : a.\, f x)}^{M} : \overbrace{a \to b}^{B}}{\underbrace{\cdot}_{\Gamma} \vdash (\lambda \underbrace{f}_{x} : \underbrace{a \to b}_{A}.\, \underbrace{\lambda x : a.\, f x}_{M}) : \underbrace{(a \to b)}_{A} \to \underbrace{a \to b}_{B}}$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\frac{\cdot\,,\ f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}{\cdot \vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\frac{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\frac{}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\dfrac{\dfrac{f : a \to b,\ x : a \vdash f x : b}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\dfrac{}{\Gamma \vdash x : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\dfrac{\dfrac{f : a \to b,\; x : a \vdash f x : b}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\dfrac{}{\Gamma \vdash x : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\dfrac{\dfrac{\dfrac{\overbrace{f : a \to b,\, x : a}^{\Gamma} \vdash \overbrace{f}^{F} : \overbrace{a}^{A} \to \overbrace{b}^{B} \qquad \overbrace{f : a \to b,\, x : a}^{\Gamma} \vdash \overbrace{x}^{M} : \overbrace{a}^{A}}{f : a \to b,\, x : a \vdash {\color{red} f x} : b}}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\dfrac{}{\Gamma \vdash x : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash {\color{red} F M} : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, fx$$

$$\dfrac{\dfrac{\dfrac{f : a \to b,\, x : a \vdash f : a \to b \qquad f : a \to b,\, x : a \vdash x : a}{f : a \to b,\, x : a \vdash fx : b}}{f : a \to b \vdash (\lambda x : a.\, fx) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, fx) : (a \to b) \to a \to b}$$

$$\dfrac{}{\Gamma \vdash x : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{f : a \to b,\, x : a \vdash {\color{red}f} : a \to b} \qquad f : a \to b,\, x : a \vdash x : a}{f : a \to b,\, x : a \vdash f x : b}}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\dfrac{}{\Gamma \vdash {\color{red}x} : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\dfrac{\dfrac{}{f : a \to b,\, x : a \vdash f : a \to b} \qquad \dfrac{}{f : a \to b,\, x : a \vdash x : a}}{\dfrac{f : a \to b,\, x : a \vdash f x : b}{\dfrac{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}}}$$

$$\dfrac{}{\Gamma \vdash x : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

example type derivation

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

$$\dfrac{\dfrac{\dfrac{\overline{f : a \to b,\, x : a \vdash f : a \to b} \qquad \overline{f : a \to b,\, x : a \vdash x : a}}{f : a \to b,\, x : a \vdash f x : b}}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\dfrac{}{\Gamma \vdash x : A} \qquad \dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \dfrac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{f : a \to b,\, x : a \vdash f : a \to b} \quad \dfrac{}{f : a \to b,\, x : a \vdash x : a}}{f : a \to b,\, x : a \vdash f x : b}}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash \color{red}{(\lambda f : a \to b.\, \lambda x : a.\, f x) :}\color{black}{} (a \to b) \to a \to b}$$

proofs versus type derivations: type derivation

$$\dfrac{\dfrac{\overline{f : a \to b,\, x : a \vdash f : a \to b} \qquad \overline{f : a \to b,\, x : a \vdash x : a}}{\dfrac{f : a \to b,\, x : a \vdash f x : b}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

proofs versus type derivations: type derivation

$$\dfrac{\dfrac{\dfrac{\overline{f : a \to b,\, x : a \vdash f : a \to b} \quad \overline{f : a \to b,\, x : a \vdash x : a}}{f : a \to b,\, x : a \vdash fx : b}}{f : a \to b \vdash (\lambda x : a.\, fx) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, fx) : (a \to b) \to a \to b}$$

$$\dfrac{\dfrac{\dfrac{\overline{a \to b,\, a \vdash a \to b} \qquad \overline{a \to b,\, a \vdash a}}{a \to b,\, a \vdash b}}{a \to b \vdash a \to b}}{\vdash (a \to b) \to a \to b}$$

$$\dfrac{\dfrac{\dfrac{\overline{f : a \to b,\ x : a \vdash f : a \to b} \quad \overline{f : a \to b,\ x : a \vdash x : a}}{f : a \to b,\ x : a \vdash f x : b}}{f : a \to b \vdash (\lambda x : a.\, f x) : a \to b}}{\vdash (\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \to b,\ a \vdash a \to b}\,\textsf{ass} \quad \dfrac{}{a \to b,\ a \vdash a}\,\textsf{ass}}{a \to b,\ a \vdash b}\,E{\to}}{a \to b \vdash a \to b}\,I{\to}}{\vdash (a \to b) \to a \to b}\,I{\to}$$

proofs versus type derivations: proof

$$\dfrac{\dfrac{\dfrac{[a \to b^{\color{red}f}] \qquad [a^{\color{red}x}]}{b}\, E{\to}}{a \to b}\, I[{\color{red}x}]{\to}}{(a \to b) \to a \to b}\, I[{\color{red}f}]{\to}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \to b,\ a \vdash a \to b}\,\textsf{ass} \qquad \dfrac{}{a \to b,\ a \vdash a}\,\textsf{ass}}{a \to b,\ a \vdash b}\, E{\to}}{a \to b \vdash a \to b}\, I{\to}}{\vdash (a \to b) \to a \to b}\, I{\to}$$

proofs versus type derivations: proof term

$$\dfrac{\dfrac{\dfrac{[a \to b^{f}] \qquad [a^{x}]}{b}\, E{\to}}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

$$\lambda f : a \to b.\, \lambda x : a.\, f x$$

Curry-Howard correspondence

propositions		types
implications	$\longleftrightarrow$	function types
proof rules		proof terms
introduction rules	$\longleftrightarrow$	lambda abstraction
elimination rules	$\longleftrightarrow$	function application
assumption rule	$\longleftrightarrow$	variables

how to read proof terms

$$M := \; (\lambda f : a \to b.\, \lambda x : a.\, f x) \; : \; (a \to b) \to a \to b$$

how to read proof terms

$$M := \ (\lambda f : a \to b.\, \lambda x : a.\, f x) \ : \ (a \to b) \to a \to b$$

this proof term M is a function
that takes as its first argument a proof of $a \to b$ called f
and as its second argument a proof of a called x

and then maps those to a proof of b

how to read proof terms

$$M := (\lambda f : a \to b.\, \lambda x : a.\, fx) \;:\; (a \to b) \to a \to b$$

this proof term M is a function
that takes as its first argument a proof of $a \to b$ called f
and as its second argument a proof of a called x

and then maps those to a proof of b

the argument f is itself a function that
maps proofs of a to proofs of b
(conform the BHK-interpretation)

x is an inhabitant of the type a
which corresponds to the proposition a

how to read proof terms

$$M := (\lambda f : a \to b.\, \lambda x : a.\, f x) \; : \; (a \to b) \to a \to b$$

this proof term M is a function
that takes as its first argument a proof of $a \to b$ called f
and as its second argument a proof of a called x

and then maps those to a proof of b

the argument f is itself a function that
maps proofs of a to proofs of b
(conform the BHK-interpretation)

x is an inhabitant of the type a
which corresponds to the proposition a

to get a proof of b, the function f is applied to x
and the result of that application then is the result of M

Coq

the example

Coq

the example

```
Parameter a b : Prop.      a is declared
                           b is declared
```

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
```

```
1 subgoal (ID 1)

  ============================
  (a -> b) -> a -> b
```

$$(a \to b) \to a \to b$$

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
```

```
1 subgoal (ID 2)

  f : a -> b
  ============================
  a -> b
```

$$\frac{a \to b}{(a \to b) \to a \to b}\ I[f]{\to}$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
```

```
1 subgoal (ID 3)

  f : a -> b
  x : a
  ============================
  b
```

$$\dfrac{\dfrac{b}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
apply f.
```

```
1 subgoal (ID 4)

  f : a -> b
  x : a
  ============================
  a
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad a}{b}\, E{\to}}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
apply f.
exact x.
```

```
No more subgoals.
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\,E{\to}}{a \to b}\,I[x]{\to}}{(a \to b) \to a \to b}\,I[f]{\to}$$

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
apply f.
exact x.
Qed.
```

$$\dfrac{\dfrac{\dfrac{[a \to b^{f}] \qquad [a^{x}]}{b}\,E{\to}}{a \to b}\,I[x]{\to}}{(a \to b) \to a \to b}\,I[f]{\to}$$

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
apply f.
exact x.
Qed.

Check one.
```

```
one
     : (a -> b) -> a -> b
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\, E{\to}}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
apply f.
exact x.
Qed.

Check one.
Print one.
```

```
one = fun (f : a -> b) (x : a) => f x
     : (a -> b) -> a -> b

Arguments one _%function_scope
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\, E\!\to}{a \to b}\, I[x]\!\to}{(a \to b) \to a \to b}\, I[f]\!\to$$

$$(\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intro f.
intro x.
apply f.
exact x.
Qed.

Check one.
Print one.
```

```
1 subgoal (ID 3)

  f : a -> b
  x : a
  ============================
  b
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\, E{\to}}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

$$(\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intros.
apply H.
exact H0.
Qed.

Check one.
Print one.
```

```
1 subgoal (ID 3)

  H : a -> b
  H0 : a
  ============================
  b
```

$$\dfrac{\dfrac{\dfrac{[a \to b^{H}] \quad [a^{H_0}]}{b}\, E{\to}}{a \to b}\, I[H_0]{\to}}{(a \to b) \to a \to b}\, I[H]{\to}$$

$$(\lambda H : a \to b.\, \lambda H_0 : a.\, H\, H_0) : (a \to b) \to a \to b$$

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intros f x.
apply f.
exact x.
Qed.

Check one.
Print one.
```

```
1 subgoal (ID 3)

  f : a -> b
  x : a
  ============================
  b
```

$$\dfrac{\dfrac{\dfrac{[a \to b^{f}] \qquad [a^{x}]}{b}\,E\!\to}{a \to b}\,I[x]\!\to}{(a \to b) \to a \to b}\,I[f]\!\to$$

$$(\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intros f x.
apply f.
exact x.
Qed.

Check one.
Print one.
```

```
No more subgoals.
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\, E{\to}}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

$$(\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intros f x.
apply f.
apply x.
Qed.

Check one.
Print one.
```

```
No more subgoals.
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\, E{\to}}{a \to b}\, I[x]{\to}}{(a \to b) \to a \to b}\, I[f]{\to}$$

$$(\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
auto.
Qed.

Check one.
Print one.
```

```
No more subgoals.
```

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
auto.
Qed.

Check one.
Print one.
```

```
one = fun H : a -> b => H
     : (a -> b) -> a -> b

Arguments one _%function_scope
```

$$\frac{[a \to b^H]}{(a \to b) \to a \to b}\, I[H]{\to}$$

$$(\lambda H : a \to b.\, H) : (a \to b) \to a \to b$$

Coq

the example

```
Parameter a b : Prop.

Lemma one :
  (a -> b) -> a -> b.
intros f x.
apply f.
apply x.
Qed.

Check one.
Print one.
```

```
one = fun (f : a -> b) (x : a) => f x
    : (a -> b) -> a -> b
```

$$\dfrac{\dfrac{\dfrac{[a \to b^f] \qquad [a^x]}{b}\,E{\to}}{a \to b}\,I[x]{\to}}{(a \to b) \to a \to b}\,I[f]{\to}$$

$$(\lambda f : a \to b.\, \lambda x : a.\, f x) : (a \to b) \to a \to b$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
```

```
1 subgoal (ID 2)

  ============================
  a \/ b -> b \/ a
```

$$a \vee b \to b \vee a$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
```

```
1 subgoal (ID 3)

  x : a \/ b
  ============================
  b \/ a
```

$$\frac{b \lor a}{a \lor b \to b \lor a}\, I[x]{\to}$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
```

```
2 subgoals (ID 4)

  x : a \/ b
  ============================
  a -> b \/ a

subgoal 2 (ID 5) is:
 b -> b \/ a
```

$$\dfrac{\dfrac{[a \vee b^x] \qquad a \to b \vee a \qquad b \to b \vee a}{b \vee a}\, E\vee}{a \vee b \to b \vee a}\, I[x]{\to}$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
```

```
2 subgoals (ID 6)

  x : a \/ b
  y : a
  ============================
  b \/ a

subgoal 2 (ID 5) is:
 b -> b \/ a
```

$$
\dfrac{[a \vee b^x] \qquad \dfrac{b \vee a}{a \to b \vee a}\, I[y]{\to} \qquad b \to b \vee a}{\dfrac{b \vee a}{a \vee b \to b \vee a}\, I[x]{\to}}\, E\vee
$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
```

```
2 subgoals (ID 8)

  x : a \/ b
  y : a
  ============================
  a

subgoal 2 (ID 5) is:
 b -> b \/ a
```

$$\dfrac{\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{a}{b \vee a}\,Ir\vee}{a \to b \vee a}\,I[y]{\to} \qquad b \to b \vee a}{b \vee a}\,E\vee}{a \vee b \to b \vee a}\,I[x]{\to}$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
apply y.
```

```
1 subgoal (ID 5)

  x : a \/ b
  ============================
  b -> b \/ a
```

$$\dfrac{\dfrac{[a \lor b^x] \qquad \dfrac{\dfrac{[a^y]}{b \lor a}\,Ir\lor}{a \to b \lor a}\,I[y]{\to} \qquad b \to b \lor a}{b \lor a}\,E\lor}{a \lor b \to b \lor a}\,I[x]{\to}$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
apply y.
intro z.
```

```
1 subgoal (ID 9)

  x : a \/ b
  z : b
  ============================
  b \/ a
```

$$\dfrac{\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\,Ir\vee}{a \to b \vee a}\,I[y]{\to} \qquad \dfrac{b \vee a}{b \to b \vee a}\,I[z]{\to}}{b \vee a}\,E\vee}{a \vee b \to b \vee a}\,I[x]{\to}$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
apply y.
intro z.
left.
```

```
1 subgoal (ID 11)

  x : a \/ b
  z : b
  ============================
  b
```

$$\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\, Ir\vee}{a \to b \vee a}\, I[y]{\to} \qquad \dfrac{\dfrac{b}{b \vee a}\, Il\vee}{b \to b \vee a}\, I[z]{\to}}{\dfrac{b \vee a}{a \vee b \to b \vee a}\, I[x]{\to}}\, E\vee$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
apply y.
intro z.
left.
apply z.
```

```
No more subgoals.
```

$$\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\,Ir\vee}{a \to b \vee a}\,I[y]{\to} \qquad \dfrac{\dfrac{[b^z]}{b \vee a}\,Il\vee}{b \to b \vee a}\,I[z]{\to}}{\dfrac{b \vee a}{a \vee b \to b \vee a}\,I[x]{\to}}\,E\vee$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
apply y.
intro z.
left.
apply z.
Qed.
```

$$\dfrac{\dfrac{[a \lor b^x] \qquad \dfrac{\dfrac{[a^y]}{b \lor a}\,Ir\lor}{a \to b \lor a}\,I[y]{\to} \qquad \dfrac{\dfrac{[b^z]}{b \lor a}\,Il\lor}{b \to b \lor a}\,I[z]{\to}}{b \lor a}\,E\lor}{a \lor b \to b \lor a}\,I[x]{\to}$$

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
intro y.
right.
apply y.
intro z.
left.
apply z.
Qed.
```

```
2 subgoals (ID 4)

  x : a \/ b
  ============================
  a -> b \/ a

subgoal 2 (ID 5) is:
 b -> b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
2 subgoals (ID 4)

  x : a \/ b
  ============================
  a -> b \/ a

subgoal 2 (ID 5) is:
 b -> b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
1 subgoal (ID 4)

  x : a \/ b
  ============================
  a -> b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
1 subgoal (ID 6)

  x : a \/ b
  y : a
  ============================
  b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
1 subgoal (ID 8)

  x : a \/ b
  y : a
  ============================
  a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
1 subgoal (ID 5)

subgoal 1 (ID 5) is:
 b -> b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
1 subgoal (ID 5)

  x : a \/ b
  ============================
  b -> b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
elim x.
- intro y.
  right.
  apply y.
- intro z.
  left.
  apply z.
Qed.
```

```
1 subgoal (ID 6)

  x : a \/ b
  y : a
  ============================
  b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
destruct x as [y|z].
- right.
  apply y.
- left.
  apply z.
Qed.
```

```
1 subgoal (ID 9)

  y : a
  ============================
  b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intro x.
destruct x as [y|z].
- right.
  apply y.
- left.
  apply z.
Qed.
```

```
1 subgoal (ID 9)

  y : a
  ============================
  b \/ a
```

example with disjunction

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intros [y|z].
- right.
  apply y.
- left.
  apply z.
Qed.
```

```
1 subgoal (ID 8)

  y : a
  ============================
  b \/ a
```

```
Parameter a b : Prop.

Lemma two :
  a \/ b -> b \/ a.
intros [y|z].
- right.
  apply y.
- left.
  apply z.
Qed.
```

$$\dfrac{\dfrac{[a \vee b^x] \qquad \dfrac{\dfrac{[a^y]}{b \vee a}\,Ir\vee}{a \to b \vee a}\,I[y]{\to} \qquad \dfrac{\dfrac{[b^z]}{b \vee a}\,Il\vee}{b \to b \vee a}\,I[z]{\to}}{b \vee a}\,E\vee}{a \vee b \to b \vee a}\,I[x]{\to}$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
```

```
1 subgoal (ID 3)

  ============================
  a /\ b -> b /\ a
```

$$a \wedge b \to b \wedge a$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
```

```
1 subgoal (ID 8)

  y : a
  z : b
  ============================
  b /\ a
```

$$
\dfrac{\dfrac{[a \land b^x] \qquad \dfrac{\dfrac{b \land a}{b \to b \land a}\,I[z]{\to}}{a \to b \to b \land a}\,I[y]{\to}}{b \land a}\,E{\land}}{a \land b \to b \land a}\,I[x]{\to}
$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
```

```
2 subgoals (ID 10)

  y : a
  z : b
  ============================
  b

subgoal 2 (ID 11) is:
 a
```

$$\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{b \qquad a}{b \wedge a}\, I\wedge}{b \to b \wedge a}\, I[z]{\to}}{a \to b \to b \wedge a}\, I[y]{\to}}{\dfrac{b \wedge a}{a \wedge b \to b \wedge a}\, I[x]{\to}}\, E\wedge$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
-
```

```
1 subgoal (ID 10)

  y : a
  z : b
  ============================
  b
```

$$\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{b \qquad a}{b \wedge a}\, I\wedge}{b \to b \wedge a}\, I[z]{\to}}{a \to b \to b \wedge a}\, I[y]{\to}}{\dfrac{b \wedge a}{a \wedge b \to b \wedge a}\, I[x]{\to}}\, E\wedge$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
- apply z.
```

```
1 subgoal (ID 11)

subgoal 1 (ID 11) is:
 a
```

$$
\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{[b^z] \qquad a}{b \wedge a}\,I\wedge}{b \to b \wedge a}\,I[z]{\to}}{a \to b \to b \wedge a}\,I[y]{\to}}{\dfrac{b \wedge a}{a \wedge b \to b \wedge a}\,I[x]{\to}}\,E\wedge
$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
- apply z.
-
```

```
1 subgoal (ID 11)

  y : a
  z : b
  ============================
  a
```

$$
\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{[b^z] \qquad a}{b \wedge a}\, I\wedge}{b \to b \wedge a}\, I[z]{\to}}{a \to b \to b \wedge a}\, I[y]{\to}}{\dfrac{b \wedge a}{a \wedge b \to b \wedge a}\, I[x]{\to}}\, E\wedge
$$

example with conjunction

```
Parameter a b : Prop.        No more subgoals.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
- apply z.
- apply y.
```

$$\dfrac{\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{[b^z] \qquad [a^y]}{b \wedge a}\, I\wedge}{b \to b \wedge a}\, I[z]{\to}}{a \to b \to b \wedge a}\, I[y]{\to}}{b \wedge a}\, E\wedge}{a \wedge b \to b \wedge a}\, I[x]{\to}$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
- apply z.
- apply y.
Qed.
```

$$\dfrac{\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{[b^z] \qquad [a^y]}{b \wedge a}\,I\wedge}{b \to b \wedge a}\,I[z]{\to}}{a \to b \to b \wedge a}\,I[y]{\to}}{b \wedge a}\,E\wedge}{a \wedge b \to b \wedge a}\,I[x]{\to}$$

example with conjunction

```
Parameter a b : Prop.

Lemma three :
  a /\ b -> b /\ a.
intros [y z].
split.
- apply z.
- apply y.
Qed.
```

$$\dfrac{\dfrac{[a \wedge b^x] \qquad \dfrac{\dfrac{\dfrac{[b^z] \qquad [a^y]}{b \wedge a}\,I\wedge}{b \to b \wedge a}\,I[z]{\to}}{a \to b \to b \wedge a}\,I[y]{\to}}{b \wedge a}\,E\wedge}{a \wedge b \to b \wedge a}\,I[x]{\to}$$

tactics for proof rules

$I{\to}$ $I\neg$	`intro` `intros`
$E{\to}$ $E\neg$	`apply`
ass	`exact` `apply`
$E\wedge$ $E\vee$ $E\bot$	`elim` `destruct` `intros` with pattern
$I\wedge$	`split`
$Il\vee$ $Ir\vee$	`left` `right`
$I\top$	`apply I`

tactics for proof rules

$I{\to}$ $I\neg$	`intro` `intros`
$E{\to}$ $E\neg$	`apply`
ass	`exact` `apply`
$E\wedge$ $E\vee$ $E\bot$	`elim` `destruct` `intros` with pattern
$I\wedge$	`split`
$Il\vee$ $Ir\vee$	`left` `right`
$I\top$	`apply I` `I : True`

bullets

structure tactic scripts according to subgoals

related bullets need to match:

-	--	---	----	etc.
+	++	+++	++++	etc.
*	**	***	****	etc.

intro patterns

only works with `intros`, not with `intro`
the pattern needs to match the shape of assumptions

goal	tactic
$A \to G$	`intros HA.`
$A \to B \to G$	`intros HA HB.`
$A \wedge B \to G$	`intros [HA HB].`
$A \vee B \to G$	`intros [HA|HB].`
$A \vee (B \wedge C) \to D \to G$	`intros [HA|[HB HC]] HD.`

apply

$$H : A_1 \to \cdots \to A_n \to B$$

$$\text{goal } B \xrightarrow{\texttt{apply H}} \text{new goals } A_1, \ldots, A_n$$

apply

the number n of antecedents of H may be zero
H not only a variable, may be an arbitrary term

$$H : A_1 \to \cdots \to A_n \to B$$

$$\text{goal } B \xrightarrow{\texttt{apply H}} \text{new goals } A_1, \ldots, A_n$$

apply

the number n of antecedents of H may be zero

H not only a variable, may be an arbitrary term

$$H : A_1 \to \cdots \to A_n \to B$$

$$\text{goal } B \xrightarrow{\texttt{apply H}} \text{new goals } A_1, \ldots, A_n$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A_1 \to \cdots \to A_n \to B^H] \qquad A_1}{A_2 \to \cdots \to A_n \to B}\,E{\to} \qquad A_2}{\ddots}\,E{\to}}{\dfrac{A_{n-1} \to A_n \to B \qquad A_{n-1}}{A_n \to B}\,E{\to} \qquad A_n}}{B}\,E{\to}$$

elim

$$H : A_1 \to \cdots \to A_n \to B \otimes C$$

`elim` and `destruct` eliminate the connective $\otimes$
which leaves the goal unchanged

but also generate extra subgoals $A_1, \ldots, A_n$

summary

minimal logic	type theory	Coq
introduction rules	lambda abstraction	`intro`
elimination rules	function application	`apply`

logic	type theory	Coq
introduction rules	lambda abstraction	`intro`
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introduction rules	(in three weeks)	(various)
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summary

logic	type theory	Coq
introduction rules	lambda abstraction	`intro`
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introduction rules	(in three weeks)	(various)
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for more about the tactics
read the Coq manual!

contents