predicate logic & dependent types

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Type Theory & Coq
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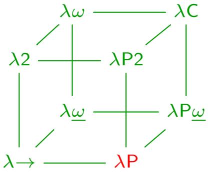

function types become dependent

for predicate logic one needs to generalize

$$(\lambda x : A.\, M) \quad : \quad A \rightarrow B$$

function types

to

$$(\lambda x : A.\, M) \quad : \quad \Pi x : A.\, B$$

dependent function types

function types become dependent

for predicate logic one needs to generalize

$$(\lambda x : A.\, M) \quad : \quad A \to B$$

function types

to

can contain x

$$(\lambda x : A.\, M) \quad : \quad \Pi x : A.\, \underset{\uparrow}{B}$$

dependent function types

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to

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$$(\lambda x : A.\, M) \quad : \quad \Pi x : A.\, \underset{\uparrow}{B}$$

dependent function types

$$\forall x : A.\, B$$

`forall x : A, B`

three different notations for the same type

we skip chapters 3 and 5 for now

inductive types

$\downarrow$

next week

program extraction

$\downarrow$

last week (briefly)

last week and this week

Curry-Howard correspondence:

propositional logic	$\lambda\!\rightarrow$	simple types
predicate logic	λP	dependent types
second order logic	$\lambda 2$	polymorphic types
beyond minimal logic	CIC	inductive types

last week and this week

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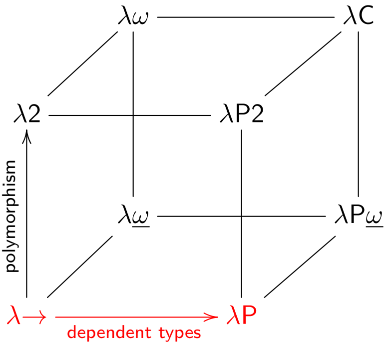
dependent types in the lambda cube



recap: propositional logic

$$A, B ::= a \mid A \to B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \bot$$

$$\begin{array}{cc} I{\to} & E{\to} \\ I\wedge & El\wedge \;\; Er\wedge \\ Il\vee \;\; Ir\vee & E\vee \\ I\neg & E\neg \\ I\top & E\bot \end{array}$$

recap: simply typed lambda calculus

$$A, B ::= a \mid A \to B$$
$$M, N ::= x \mid M\,N \mid \lambda x : A.\, M$$

$$\frac{}{\Gamma \vdash x : A} \qquad \text{for } (x : A) \in \Gamma$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.\, M) : A \to B} \qquad \frac{\Gamma \vdash F : A \to B \qquad \Gamma \vdash M : A}{\Gamma \vdash F M : B}$$

three predicate logics

- minimal predicate logic

$$\to \quad \forall$$

- constructive predicate logic

$$\to \quad \wedge \quad \vee \quad \neg \quad \top \quad \bot \quad \forall \quad \exists$$

- classical predicate logic

$$A \vee \neg A$$
$$\neg\neg A \to A$$

propositional versus predicate logic: syntax

$$A, B ::= a \mid A \to B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \bot$$

$$\begin{aligned} M, N &::= x \mid f(\overrightarrow{M}) \\ \overrightarrow{M} &::= \cdot \mid \overrightarrow{M}, N \\ A, B &::= p(\overrightarrow{M}) \mid A \to B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \bot \mid \\ &\qquad \forall x.\, A \mid \exists x.\, A \end{aligned}$$

$\forall$ and $\exists$ bind weakly

$$\dfrac{\begin{matrix}[A^H] \\ \vdots \\ B\end{matrix}}{A \to B}\; I[H]{\to} \qquad \dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\; E{\to}$$

$$\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall x.\, A}\; I\forall \qquad \dfrac{\begin{matrix}\vdots \\ \forall x.\, A\end{matrix}}{A[x := M]}\; E\forall$$

variable condition of $I\forall$: x not free in available assumptions

same four rules with explicit assumptions

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\, I{\to} \qquad \frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\, E{\to}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x.\, A}\, I\forall \qquad \frac{\Gamma \vdash \forall x.\, A}{\Gamma \vdash A[x := M]}\, E\forall$$

variable condition of $I\forall$: x not free in Γ

rules for the existential quantifier

$$\dfrac{\begin{array}{c}\vdots\\ A[x := M]\end{array}}{\exists x.\, A}\ I\exists \qquad\qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \exists x.\, A & \forall x.\, A \to C\end{array}}{C}\ E\exists$$

variable condition of $E\exists$: x not free in C

example proof in minimal logic

$$\forall x. \big((\forall y.\, p(y)) \to p(x) \big)$$

$$\forall x. \big((\forall y.\, p(y)) \to p(x)\big)$$

$$\cfrac{\begin{matrix}[A^H] \\ \vdots \\ B\end{matrix}}{A \to B}\, I[H]{\to} \qquad \cfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall x.\, A}\, I\forall \qquad \cfrac{\begin{matrix}\vdots \\ \forall x.\, A\end{matrix}}{A[x := M]}\, E\forall$$

example proof in minimal logic

$$\frac{(\forall y.\, p(y)) \to p(x)}{\forall x. \big((\forall y.\, p(y)) \to p(x)\big)}\; I\forall$$

$$\begin{array}{c}[A^H] \\ \vdots \\ \dfrac{B}{A \to B}\; I[H]{\to}\end{array} \qquad \begin{array}{c}\vdots \\ \dfrac{A}{\forall x.\, A}\; I\forall\end{array} \qquad \begin{array}{c}\vdots \\ \dfrac{\forall x.\, A}{A[x := M]}\; E\forall\end{array}$$

example proof in minimal logic

$[\forall y.\, p(y)^H]$

$$\dfrac{\dfrac{p(x)}{(\forall y.\, p(y)) \to p(x)}\; I[H]{\to}}{\forall x.\big((\forall y.\, p(y)) \to p(x)\big)}\; I\forall$$

$$\dfrac{\begin{array}{c}[A^H]\\ \vdots \\ B\end{array}}{A \to B}\; I[H]{\to} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array}}{\forall x.\, A}\; I\forall \qquad \dfrac{\begin{array}{c}\vdots \\ \forall x.\, A\end{array}}{A[x := M]}\; E\forall$$

example proof in minimal logic

$[\forall y.\, p(y)^H]$

$$\dfrac{\dfrac{\dfrac{\forall y.\, p(y)}{p(x)}\, E\forall}{(\forall y.\, p(y)) \to p(x)}\, I[H]{\to}}{\forall x.\big((\forall y.\, p(y)) \to p(x)\big)}\, I\forall$$

$$\dfrac{\begin{matrix}[A^H] \\ \vdots \\ B\end{matrix}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall x.\, A}\, I\forall \qquad \dfrac{\begin{matrix}\vdots \\ \forall x.\, A\end{matrix}}{A[x := M]}\, E\forall$$

example proof in minimal logic

$[\forall y.\, p(y)^H]$

$$\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\, E\forall}{(\forall y.\, p(y)) \to p(x)}\, I[H]{\to}}{\forall x.\big((\forall y.\, p(y)) \to p(x)\big)}\, I\forall$$

$$\dfrac{\begin{array}{c}[A^H] \\ \vdots \\ B\end{array}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array}}{\forall x.\, A}\, I\forall \qquad \dfrac{\begin{array}{c}\vdots \\ \forall x.\, A\end{array}}{A[x := M]}\, E\forall$$

example proof in minimal logic

$[\forall y.\, p(y)^H]$

$$\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\; E\forall}{(\forall y.\, p(y)) \to p(x)}\; I[H]{\to}}{\forall x.\big((\forall y.\, p(y)) \to p(x)\big)}\; I\forall$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions (there are no available assumptions at that point)

$$\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\, E\forall}{(\forall y.\, p(y)) \to p(x)}\, I[H]{\to}}{\forall x.\big((\forall y.\, p(y)) \to p(x)\big)}\, I\forall$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions (there are no available assumptions at that point)

example with existential quantifier

$$(\exists x.\, p(x)) \leftrightarrow \neg(\forall x.\, \neg p(x))$$

example with existential quantifier

$A \leftrightarrow B := (A \to B) \land (B \to A)$

$$(\exists x.\, p(x)) \leftrightarrow \neg(\forall x.\, \neg p(x))$$

$$(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))$$
$$\neg(\forall x.\, \neg p(x)) \to (\exists x.\, p(x))$$

example with existential quantifier

$A \leftrightarrow B := (A \to B) \land (B \to A)$

$$(\exists x.\, p(x)) \leftrightarrow \neg(\forall x.\, \neg p(x))$$

not constructively valid

$$(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))$$
$$\neg(\forall x.\, \neg p(x)) \to (\exists x.\, p(x))$$

example with existential quantifier

$A \leftrightarrow B := (A \to B) \wedge (B \to A)$

$$(\exists x.\, p(x)) \leftrightarrow \neg(\forall x.\, \neg p(x))$$

not constructively valid

$$\begin{aligned} (\exists x.\, p(x)) &\to \neg(\forall x.\, \neg p(x)) \\ \neg(\forall x.\, \neg p(x)) &\to \neg\neg(\exists x.\, p(x)) \end{aligned}$$

example from left to right

$$(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))$$

$$(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))$$

$$\dfrac{\begin{matrix}[A^H]\\ \vdots\\ B\end{matrix}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{matrix}[A^H]\\ \vdots\\ \bot\end{matrix}}{\neg A}\, I[H]\neg \qquad \dfrac{\begin{matrix}\vdots\\ \exists x.\, A\end{matrix} \qquad \begin{matrix}\vdots\\ \forall x.\, A \to C\end{matrix}}{C}\, E\exists$$

example from left to right

$[\exists x.\, p(x)^{H_0}]$

$$\frac{\neg(\forall x.\, \neg p(x))}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\; I[H_0]{\to}$$

$$\frac{\begin{array}{c}[A^H]\\ \vdots \\ B\end{array}}{A \to B}\; I[H]{\to} \qquad \frac{\begin{array}{c}[A^H]\\ \vdots \\ \bot\end{array}}{\neg A}\; I[H]\neg \qquad \frac{\begin{array}{c}\vdots \\ \exists x.\, A\end{array} \quad \begin{array}{c}\vdots \\ \forall x.\, A \to C\end{array}}{C}\; E\exists$$

example from left to right

$[\exists x.\, p(x)^{H_0}]$ $[\forall x.\, \neg p(x)^{H_1}]$

$$\dfrac{\dfrac{\bot}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

$$\dfrac{\begin{array}{c}[A^H]\\ \vdots\\ B\end{array}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{array}{c}[A^H]\\ \vdots\\ \bot\end{array}}{\neg A}\, I[H]\neg \qquad \dfrac{\begin{array}{c}\vdots\\ \exists x.\, A\end{array} \quad \begin{array}{c}\vdots\\ \forall x.\, A \to C\end{array}}{C}\, E\exists$$

example from left to right

$[\exists x.\, p(x)^{H_0}]$ $[\forall x.\, \neg p(x)^{H_1}]$

$$\dfrac{\dfrac{\dfrac{\exists x.\, p(x) \qquad \forall x.\, p(x) \to \bot}{\bot}\, E\exists}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

$$\dfrac{\begin{matrix}[A^H]\\ \vdots \\ B\end{matrix}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{matrix}[A^H]\\ \vdots \\ \bot\end{matrix}}{\neg A}\, I[H]\neg \qquad \dfrac{\begin{matrix}\vdots \\ \exists x.\, A\end{matrix} \qquad \begin{matrix}\vdots \\ \forall x.\, A \to C\end{matrix}}{C}\, E\exists$$

example from left to right

$[\exists x.\, p(x)^{H_0}]$ $[\forall x.\, \neg p(x)^{H_1}]$

$$\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad \forall x.\, p(x) \to \bot}{\bot}\, E\exists}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

$$\dfrac{\begin{array}{c}[A^H]\\ \vdots \\ B\end{array}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{array}{c}[A^H]\\ \vdots \\ \bot\end{array}}{\neg A}\, I[H]\neg \qquad \dfrac{\begin{array}{c}\vdots \\ \exists x.\, A\end{array} \qquad \begin{array}{c}\vdots \\ \forall x.\, A \to C\end{array}}{C}\, E\exists$$

example from left to right

$[\exists x.\, p(x)^{H_0}]$ $[\forall x.\, \neg p(x)^{H_1}]$

$$\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \quad [\forall x.\, p(x) \to \bot^{H_1}]}{\bot}\, E\exists}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

$$\dfrac{\begin{matrix}[A^H] \\ \vdots \\ B\end{matrix}}{A \to B}\, I[H]{\to} \qquad \dfrac{\begin{matrix}[A^H] \\ \vdots \\ \bot\end{matrix}}{\neg A}\, I[H]\neg \qquad \dfrac{\begin{matrix}\vdots \\ \exists x.\, A\end{matrix} \quad \begin{matrix}\vdots \\ \forall x.\, A \to C\end{matrix}}{C}\, E\exists$$

example from left to right

$$\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad [\forall x.\, p(x) \to \bot^{H_1}]}{\bot}\, E\exists}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

variable condition check:

at the $E\exists$ step the variable x does not occur in $\bot$

example from left to right

$$
\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad [\forall x.\, p(x) \to \bot^{H_1}]}{\bot}\, E\exists}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}
$$

variable condition check:

at the $E\exists$ step the variable x does not occur in $\bot$

example from right to left

$$\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$

$$\dfrac{\neg\neg(\exists x.\, p(x))}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\; I[H_0]{\to}$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$ $[\neg(\exists x.\, p(x))^{H_1}]$

$$\dfrac{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\, I[H_0]{\to}$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]\ \ [\neg(\exists x.\, p(x))^{H_1}]$

$$\dfrac{\dfrac{\neg(\forall x.\, \neg p(x)) \qquad \forall x.\, \neg p(x)}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\ I[H_1]\neg}\ E\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\ I[H_0]{\to}$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$ $[\neg(\exists x.\, p(x))^{H_1}]$

$$\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \forall x.\, \neg p(x)}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}\, E\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\, I[H_0]\!\to$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]\ \ [\neg(\exists x.\, p(x))^{H_1}]$

$$
\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\neg p(x)}{\forall x.\, \neg p(x)}\, I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}\, E\neg}{\neg(\forall x.\, \neg p(x)) \rightarrow \neg\neg(\exists x.\, p(x))}\, I[H_0]\!\rightarrow
$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]\ \ [\neg(\exists x.\, p(x))^{H_1}]\ \ [p(x)^{H_2}]$

$$
\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\bot}{\neg p(x)}\, I[H_2]\neg}{\forall x.\, \neg p(x)}\, I\forall}{\dfrac{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\, I[H_0]\!\to}\, E\neg
$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$ $[\neg(\exists x.\, p(x))^{H_1}]$ $[p(x)^{H_2}]$

$$\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{\neg(\exists x.\, p(x)) \qquad \exists x.\, p(x)}{\bot}\, E{\rightarrow}}{\neg p(x)}\, I[H_2]\neg}{\forall x.\, \neg p(x)}\, I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}\, E\neg}{\neg(\forall x.\, \neg p(x)) \rightarrow \neg\neg(\exists x.\, p(x))}\, I[H_0]{\rightarrow}$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$ $[\neg(\exists x.\, p(x))^{H_1}]$ $[p(x)^{H_2}]$

$$
\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\, p(x))^{H_1}] \qquad \exists x.\, p(x)}{\bot}\, E{\rightarrow}}{\neg p(x)}\, I[H_2]\neg}{\forall x.\, \neg p(x)}\, I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}\, E\neg}{\neg(\forall x.\, \neg p(x)) \rightarrow \neg\neg(\exists x.\, p(x))}\, I[H_0]{\rightarrow}
$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]\ \ [\neg(\exists x.\, p(x))^{H_1}]\ \ [p(x)^{H_2}]$

$$
\begin{prooftree}
\AxiomC{$[\neg(\exists x.\, p(x))^{H_1}]$}
\AxiomC{$p(x)$}
\RightLabel{$I\exists$}
\UnaryInfC{$\exists x.\, p(x)$}
\RightLabel{$E{\to}$}
\BinaryInfC{$\bot$}
\RightLabel{$I[H_2]\neg$}
\UnaryInfC{$\neg p(x)$}
\RightLabel{$I\forall$}
\UnaryInfC{$\forall x.\, \neg p(x)$}
\AxiomC{$[\neg(\forall x.\, \neg p(x))^{H_0}]$}
\RightLabel{$E\neg$}
\BinaryInfC{$\bot$}
\RightLabel{$I[H_1]\neg$}
\UnaryInfC{$\neg\neg(\exists x.\, p(x))$}
\RightLabel{$I[H_0]{\to}$}
\UnaryInfC{$\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))$}
\end{prooftree}
$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$ $[\neg(\exists x.\, p(x))^{H_1}]$ $[p(x)^{H_2}]$

$$\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\, p(x))^{H_1}] \qquad \dfrac{[p(x)^{H_2}]}{\exists x.\, p(x)}\, I\exists}{\bot}\, E{\rightarrow}}{\neg p(x)}\, I[H_2]\neg}{\forall x.\, \neg p(x)}\, I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}\, E\neg}{\neg(\forall x.\, \neg p(x)) \rightarrow \neg\neg(\exists x.\, p(x))}\, I[H_0]{\rightarrow}$$

example from right to left

$[\neg(\forall x.\, \neg p(x))^{H_0}]$ $[\neg(\exists x.\, p(x))^{H_1}]$ $[p(x)^{H_2}]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\, p(x))^{H_1}] \qquad \dfrac{[p(x)^{H_2}]}{\exists x.\, p(x)}\; I\exists}{\bot}\; E{\to}}{\neg p(x)}\; I[H_2]\neg}{\forall x.\, \neg p(x)}\; I\forall}{\bot}\; E\neg}{\neg\neg(\exists x.\, p(x))}\; I[H_1]\neg}{}}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\; I[H_0]{\to}
$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions

example from right to left

$$\dfrac{\dfrac{\dfrac{[\neg(\forall x.\,\neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\,p(x))^{H_1}] \qquad \dfrac{[p(x)^{H_2}]}{\exists x.\,p(x)}\,I\exists}{\bot}\,E{\to}}{\neg p(x)}\,I[H_2]\neg}{\forall x.\,\neg p(x)}\,I\forall}{\bot}\,E\neg}{\neg\neg(\exists x.\,p(x))}\,I[H_1]\neg}{\neg(\forall x.\,\neg p(x)) \to \neg\neg(\exists x.\,p(x))}\,I[H_0]{\to}$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions

types thus far

- atomic types
  $a, b, c, \ldots$
- types of proof objects of propositions
  $a, b, c, \ldots$
- function types
- datatypes
  - natural numbers
  - Booleans
  - lists
  - binary trees
  - ...

how are types introduced?

- free type variables

  STT = simple type theory

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- free type variables

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- in the context

  PTSs = pure type systems: $\lambda{\rightarrow}$ λP $\lambda 2$ λC

how are types introduced?

- free type variables

  STT = simple type theory

- in the context

  PTSs = pure type systems: $\lambda\!\rightarrow$ λP λ2 λC

$$\Gamma \vdash M : A$$

$$\mathsf{O} : \mathsf{nat},\ \mathsf{S} : \mathsf{nat} \to \mathsf{nat} \vdash \mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) : \mathsf{nat}$$

how are types introduced?

- free type variables

  STT = simple type theory

- in the context

  PTSs = pure type systems: $\lambda{\to}$ λP λ2 λC

$$\Gamma \vdash M : A$$

$$\mathsf{nat} : *,\ \mathsf{O} : \mathsf{nat},\ \mathsf{S} : \mathsf{nat} \to \mathsf{nat} \vdash \mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) : \mathsf{nat}$$

how are types introduced?

- free type variables

  STT = simple type theory

- in the context = 'axiomatic'

  PTSs = pure type systems: $\lambda\!\rightarrow$ λP λ2 λC

$$\Gamma \vdash M : A$$

$$\mathsf{nat} : *,\ \mathsf{O} : \mathsf{nat},\ \mathsf{S} : \mathsf{nat} \to \mathsf{nat} \vdash \mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) : \mathsf{nat}$$

Coq: `Parameter`

how are types introduced?

- free type variables

  STT = simple type theory

- in the context = 'axiomatic'

  PTSs = pure type systems: $\lambda\!\rightarrow$ λP λ2 λC

  $$\Gamma \vdash M : A$$

  $$\mathsf{nat} : *,\ \mathsf{O} : \mathsf{nat},\ \mathsf{S} : \mathsf{nat} \to \mathsf{nat} \vdash \mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) : \mathsf{nat}$$

  Coq: `Parameter`

- definitions ⟶ next week

  CIC = Calculus of Inductive Constructions

  $$E[\Gamma] \vdash M : A$$

  Coq: `Definition` `Inductive`

star and box

the sorts of the lambda cube:

$$\begin{aligned} * &= \text{the type of types} \\ \Box &= \text{the type of } * \end{aligned}$$

star and box

the sorts of the lambda cube:

$$\begin{aligned} * &= \text{the type of types} \\ \Box &= \text{the type of } * \end{aligned}$$

the sort $*$ is a kind
$=$ third of four levels

$$\text{object} : \text{type}$$
$$\text{type} : \text{kind}$$
$$\text{kind} : \Box$$

$$\text{object} : \text{type} : \text{kind} : \Box$$
$$\mathsf{S} : \text{nat} \to \text{nat} : * : \Box$$

example contexts

natural numbers:

$$\begin{aligned} \mathsf{nat} &: * \\ \mathsf{O} &: \mathsf{nat} \\ \mathsf{S} &: \mathsf{nat} \to \mathsf{nat} \\ \mathsf{add} &: \mathsf{nat} \to \mathsf{nat} \to \mathsf{nat} \end{aligned}$$

Booleans:

$$\begin{aligned} \mathsf{bool} &: * \\ \mathsf{true} &: \mathsf{bool} \\ \mathsf{false} &: \mathsf{bool} \end{aligned}$$

lists

lists of Booleans:

$$\begin{aligned}
\mathsf{list} &: * \\
\mathsf{nil} &: \mathsf{list} \\
\mathsf{cons} &: \mathsf{bool} \to \mathsf{list} \to \mathsf{list} \\
\mathsf{hd} &: \mathsf{list} \to \mathsf{bool} \\
\mathsf{tl} &: \mathsf{list} \to \mathsf{list} \\
\mathsf{append} &: \mathsf{list} \to \mathsf{list} \to \mathsf{list} \\
\mathsf{reverse} &: \mathsf{list} \to \mathsf{list}
\end{aligned}$$

lists of Booleans:

$$
\begin{aligned}
\mathsf{list} &: * \\
\mathsf{nil} &: \mathsf{list} \\
\mathsf{cons} &: \mathsf{bool} \to \mathsf{list} \to \mathsf{list} \\
\mathsf{hd} &: \mathsf{list} \to \mathsf{bool} \\
\mathsf{tl} &: \mathsf{list} \to \mathsf{list} \\
\mathsf{append} &: \mathsf{list} \to \mathsf{list} \to \mathsf{list} \\
\mathsf{reverse} &: \mathsf{list} \to \mathsf{list}
\end{aligned}
$$

cons has two arguments
(the head and tail to be put together)

trueslist

the function 'trueslist' maps a number n to a list of length n consisting of n copies of 'true'

$$
\begin{aligned}
\mathsf{trueslist}\ \mathsf{O} &= \mathsf{nil} \\
\mathsf{trueslist}\ (\mathsf{S}\ \mathsf{O}) &= \mathsf{cons\ true\ nil} \\
\mathsf{trueslist}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) &= \mathsf{cons\ true}\ (\mathsf{cons\ true\ nil}) \\
\mathsf{trueslist}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O}))) &= \mathsf{cons\ true}\ (\mathsf{cons\ true}\ (\mathsf{cons\ true\ nil})) \\
&\ldots
\end{aligned}
$$

$$\mathsf{trueslist} \ : \ \mathsf{nat} \to \mathsf{list}$$

vectors and truesvec

vectors of Booleans:

$$\mathsf{vec} : \mathsf{nat} \to *$$

$\mathsf{vec}\ \mathsf{O} : *$	vectors of length 0
$\mathsf{vec}\ (\mathsf{S}\ \mathsf{O}) : *$	vectors of length 1
$\mathsf{vec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) : *$	vectors of length 2

$$\ldots$$

vectors of Booleans:

$$\begin{aligned} \mathsf{vec} &: \mathsf{nat} \to * \\ \mathsf{vec}\ \mathsf{O} &: * && \text{vectors of length 0} \\ \mathsf{vec}\ (\mathsf{S}\ \mathsf{O}) &: * && \text{vectors of length 1} \\ \mathsf{vec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) &: * && \text{vectors of length 2} \\ &\dots \end{aligned}$$

$$\mathsf{trueslist} : \mathsf{nat} \to \mathsf{list}$$

$$\mathsf{truesvec} : \mathsf{nat} \to \mathsf{vec}$$

vectors and truesvec

vectors of Booleans:

$$\begin{aligned} \mathsf{vec} &: \mathsf{nat} \to * \\ \mathsf{vec}\ \mathsf{O} &: * && \text{vectors of length 0} \\ \mathsf{vec}\ (\mathsf{S}\ \mathsf{O}) &: * && \text{vectors of length 1} \\ \mathsf{vec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) &: * && \text{vectors of length 2} \\ &\cdots \end{aligned}$$

$$\mathsf{trueslist} : \mathsf{nat} \to \mathsf{list}$$

$$\mathsf{truesvec} : \mathsf{nat} \to \mathsf{vec}\ n$$

vectors and truesvec

vectors of Booleans:

$$\mathsf{vec} : \mathsf{nat} \to *$$

$\mathsf{vec}\ \mathsf{O} : *$	vectors of length 0
$\mathsf{vec}\ (\mathsf{S}\ \mathsf{O}) : *$	vectors of length 1
$\mathsf{vec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) : *$	vectors of length 2
$\ldots$	

$$\mathsf{trueslist} : \mathsf{nat} \to \mathsf{list}$$

$$\mathsf{truesvec} : \mathsf{nat} \to \mathsf{vec}\ n$$

$$\mathsf{truesvec} : \Pi n : \mathsf{nat}.\, \mathsf{vec}\ n$$

vnil and vcons

$$\mathsf{vnil} : \mathsf{vec}\ \mathsf{O}$$

$$\mathsf{vcons} : \mathsf{bool} \to \mathsf{vec}\ n \to \mathsf{vec}\ (\mathsf{S}\ n)$$

vnil and vcons

$$\text{vnil} : \text{vec } \mathsf{O}$$

$$\text{vcons} : \text{nat} \to \text{bool} \to \text{vec } n \to \text{vec } (\mathsf{S}\ n)$$

vnil and vcons

$$\text{vnil} : \text{vec } \mathsf{O}$$

$$\text{vcons} : \text{nat} \rightarrow \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (\mathsf{S}\ n)$$

$$\text{vcons} : \Pi n : \text{nat}.\ \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (\mathsf{S}\ n)$$

vnil and vcons

$$\mathsf{vnil} : \mathsf{vec}\ \mathsf{O}$$

$$\mathsf{vcons} : \mathsf{nat} \to \mathsf{bool} \to \mathsf{vec}\ n \to \mathsf{vec}\ (\mathsf{S}\ n)$$

$$\mathsf{vcons} : \Pi n : \mathsf{nat}.\ \mathsf{bool} \to \mathsf{vec}\ n \to \mathsf{vec}\ (\mathsf{S}\ n)$$

vcons has three arguments!

$$\mathsf{vnil} : \mathsf{vec}\ \mathsf{O}$$

$$\mathsf{vcons} : \mathsf{nat} \to \mathsf{bool} \to \mathsf{vec}\ n \to \mathsf{vec}\ (\mathsf{S}\ n)$$

$$\mathsf{vcons} : \Pi n : \mathsf{nat}.\ \mathsf{bool} \to \mathsf{vec}\ n \to \mathsf{vec}\ (\mathsf{S}\ n)$$

vcons has three arguments!

$$\mathsf{truesvec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O}))\ :\ \mathsf{vec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O}))$$

$$\mathsf{truesvec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) = \mathsf{vcons}\ (\mathsf{S}\ \mathsf{O})\ \mathsf{true}\ (\mathsf{vcons}\ \mathsf{O}\ \mathsf{true}\ \mathsf{vnil})$$

types when applying vcons

$$\mathsf{truesvec}\ (\mathsf{S}\ \mathsf{O}) = \mathsf{vcons}\ \mathsf{O}\ \mathsf{true}\ \mathsf{vnil}$$

$$\begin{aligned}
\mathsf{truesvec} &: \Pi n : \mathsf{nat}.\, \mathsf{vec}\ n \\
\mathsf{truesvec}\ (\mathsf{S}\ \mathsf{O}) &: \mathsf{vec}\ (\mathsf{S}\ \mathsf{O})
\end{aligned}$$

$$\begin{aligned}
\mathsf{vcons} &: \Pi n : \mathsf{nat}.\, \mathsf{bool} \to \mathsf{vec}\ n \to \mathsf{vec}\ (\mathsf{S}\ n) \\
\mathsf{O} &: \mathsf{nat} \\
\mathsf{vcons}\ \mathsf{O} &: \mathsf{bool} \to \mathsf{vec}\ \mathsf{O} \to \mathsf{vec}\ (\mathsf{S}\ \mathsf{O}) \\
\mathsf{true} &: \mathsf{bool} \\
\mathsf{vcons}\ \mathsf{O}\ \mathsf{true} &: \mathsf{vec}\ \mathsf{O} \to \mathsf{vec}\ (\mathsf{S}\ \mathsf{O}) \\
\mathsf{vnil} &: \mathsf{vec}\ \mathsf{O} \\
\mathsf{vcons}\ \mathsf{O}\ \mathsf{true}\ \mathsf{vnil} &: \mathsf{vec}\ (\mathsf{S}\ \mathsf{O})
\end{aligned}$$

Coq

BHK-interpretation

proof of $A \to B$	function from proofs of A to proofs of B
proof of $\neg A$	function from proofs of A to proofs of $\bot$
proof of $A \wedge B$	pair of a proof of A and a proof of B
proof of $A \vee B$	either a proof of A and a proof of B

Coq

BHK-interpretation

proof of $A \to B$	function from proofs of A to proofs of B
proof of $\neg A$	function from proofs of A to proofs of $\bot$
proof of $A \wedge B$	pair of a proof of A and a proof of B
proof of $A \vee B$	either a proof of A and a proof of B
proof of $\forall x.\, A$	dependent function from objects to proofs of A
proof of $\exists x.\, A$	dependent pair of an object and a proof of A

dependent $=$ A is not fixed but *depends* on the object x

Coq

BHK-interpretation

proof of $A \to B$	function from proofs of A to proofs of B
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proof of $\forall x.\, A$	dependent function from objects to proofs of A
proof of $\exists x.\, A$	dependent pair of an object and a proof of A

dependent $=$ A is not fixed but *depends* on the object x

$$A \vee B \;\leftrightarrow\; \exists x \in \{\text{left}, \text{right}\}.\, (x = \text{left} \to A) \wedge (x = \text{right} \to B)$$

Curry-Howard correspondence

type theory	Coq	minimal logic
$A \to B$	A `->` B	$A \to B$
$\Pi x : D.\, A$	`forall x :` D`,` A	$\forall x.\, A$

Curry-Howard correspondence

type theory	Coq	minimal logic
$A \to B$	A `->` B	$A \to B$
$\Pi x : D.\, A$	`forall x :` D`,` A	$\forall x.\, A$

Curry-Howard correspondence

type theory	Coq	minimal logic
$A \to B$	`A -> B`	$A \to B$
$\Pi x : D.\, A$	`forall x : D, A`	$\forall x.\, A$

$D : *$ is called `Terms` in the notes

Curry-Howard correspondence

type theory	Coq	minimal logic
$A \to B$	A `->` B	$A \to B$
$\Pi x : D.\, A$	`forall x :` D`,` A	$\forall x.\, A$

$D : *$ is called `Terms` in the notes

proof term		type	statement proved
H	:	$A \to B$	$A \to B$
H'	:	A	A
$H H'$	:	B	B

Curry-Howard correspondence

type theory	Coq	minimal logic
$A \to B$	A `->` B	$A \to B$
$\Pi x : D.\, A$	`forall x :` D`,` A	$\forall x.\, A$

$D : *$ is called `Terms` in the notes

proof term		type	statement proved
H	:	$A \to B$	$A \to B$
H'	:	A	A
$H\,H'$	:	B	B
H	:	$\Pi x : D.\, P x$	$\forall x.\, P(x)$
$H\,M$	:	$P\,M$	$P(M)$

Coq version of the first example

```
Parameter D : Set.            D is declared
```

Coq version of the first example

```
Parameter D : Set.              p is declared
Parameter p : D -> Prop.
```

Coq version of the first example

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma four :
  forall x : D,
  (forall y : D, p y) ->
  p x.
```

```
1 subgoal (ID 1)

  ============================
  forall x : D,
  (forall y : D, p y) -> p x
```

$$\forall x. (\forall y. p(y)) \to p(x)$$

Coq version of the first example

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma four :
  forall x : D,
  (forall y : D, p y) ->
  p x.
intros x H.
```

```
1 subgoal (ID 3)

  x : D
  H : forall y : D, p y
  ============================
  p x
```

$$\dfrac{\dfrac{p(x)}{(\forall y.\, p(y)) \to p(x)}\; I[H]{\to}}{\forall x.\, (\forall y.\, p(y)) \to p(x)}\; I\forall$$

Coq version of the first example

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma four :
  forall x : D,
  (forall y : D, p y) ->
  p x.
intros x H.
apply H.
```

```
No more subgoals.
```

$$
\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\,E\forall}{(\forall y.\, p(y)) \to p(x)}\,I[H]{\to}}{\forall x.\, (\forall y.\, p(y)) \to p(x)}\,I\forall
$$

Coq version of the first example

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma four :
  forall x : D,
  (forall y : D, p y) ->
  p x.
intros x H.
apply H.
Qed.
```

$$\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\,E\forall}{(\forall y.\, p(y)) \to p(x)}\,I[H]{\to}}{\forall x.\, (\forall y.\, p(y)) \to p(x)}\,I\forall$$

Coq version of the first example

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma four :
  forall x : D,
  (forall y : D, p y) ->
  p x.
intros x H.
apply H.
Qed.

Print four.
```

```
four =
fun (x : D) (H : forall y : D, p y)
=> H x
     : forall x : D,
       (forall y : D, p y) -> p x

Arguments four _ _%function_scope
```

$$\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\, E\forall}{(\forall y.\, p(y)) \to p(x)}\, I[H]{\to}}{\forall x.\, (\forall y.\, p(y)) \to p(x)}\, I\forall$$

$$\big(\lambda x : D.\, \lambda H : (\Pi y : D.\, py).\, Hx\big) : \Pi x : D.\, (\Pi y : D.\, py) \to px$$

Coq version of the first example

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma four :
  forall x : D,
  (forall y : D, p y) ->
  p x.
intros x H.
apply H.
Qed.

Print four.
```

```
four =
fun (x : D) (H : forall y : D, p y)
=> H x
     : forall x : D,
       (forall y : D, p y) -> p x

Arguments four _ _%function_scope
```

$$
\dfrac{\dfrac{\dfrac{[\forall y.\, p(y)^H]}{p(x)}\, E\forall}{(\forall y.\, p(y)) \to p(x)}\, I[H]{\to}}{\forall x.\, (\forall y.\, p(y)) \to p(x)}\, I\forall
$$

$$
(\lambda x : D.\, \lambda H : (\Pi y : D.\, py).\, Hx) : \Pi x : D.\, (\Pi y : D.\, py) \to px
$$

understanding the proof term

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

understanding the proof term

$$\forall x.\,(\forall y.\,p(y)) \to p(x)$$

$$\lambda x : D.$$

understanding the proof term

$$\forall x.\, (\forall y.\, p(y)) \to p(x)$$

$$\lambda x : D.\, \lambda H : \underbrace{(\Pi y : D.\, p y)}_{\forall y.\, p(y)}.$$

understanding the proof term

$$\forall x.\,(\forall y.\,p(y)) \to p(x)$$

$$\lambda x : D.\, \lambda H : (\Pi y : D.\, py).\, \underbrace{Hx}_{:\, px}$$

understanding the proof term

$$\forall x.\, (\forall y.\, p(y)) \to p(x)$$

$$\lambda x : D.\, \lambda H : (\Pi y : D.\, py).\, Hx$$

the second example in Coq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
```

```
1 subgoal (ID 3)

  ============================
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x)
```

$$(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))$$

the second example in Coq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
intros H0 H1.
```

```
1 subgoal (ID 6)

  H0 : exists x : D, p x
  H1 : forall x : D, ~ p x
  ============================
  False
```

$$\dfrac{\dfrac{\bot}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

the second example in Coq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
intros H0 H1.
elim H0.
```

```
1 subgoal (ID 7)

  H0 : exists x : D, p x
  H1 : forall x : D, ~ p x
  ============================
  forall x : D, p x -> False
```

$$\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad \forall x.\, p(x) \to \bot}{\bot}\; E\exists}{\neg(\forall x.\, \neg p(x))}\; I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\; I[H_0]{\to}$$

the second example in Coq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
intros H0 H1.
elim H0.
apply H1.
```

```
No more subgoals.
```

$$\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad [\forall x.\, p(x) \to \bot^{H_1}]}{\bot}\, E\exists}{\neg(\forall x.\, \neg p(x))}\, I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\, I[H_0]{\to}$$

the second example in Coq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
intros H0 H1.
elim H0.
apply H1.
Qed.
```

$$\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad [\forall x.\, p(x) \to \bot^{H_1}]}{\dfrac{\bot}{\neg(\forall x.\, \neg p(x))}\; I[H_1]\neg}\; E\exists}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\; I[H_0]{\to}$$

the second example in Coq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
intros H0 H1.
elim H0.
apply H1.
Qed.
```

$$\dfrac{\dfrac{\dfrac{[\exists x.\, p(x)^{H_0}] \qquad [\forall x.\, p(x) \to \bot^{H_1}]}{\bot}\; E\exists}{\neg(\forall x.\, \neg p(x))}\; I[H_1]\neg}{(\exists x.\, p(x)) \to \neg(\forall x.\, \neg p(x))}\; I[H_0]{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
```

```
1 subgoal (ID 5)

  ============================
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x)
```

$$\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
```

```
1 subgoal (ID 8)

  H0 : ~ (forall x : D, ~ p x)
  H1 : ~ (exists x : D, p x)
  ============================
  False
```

$$\dfrac{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\; I[H_1]^{\neg}}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\; I[H_0]^{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
```

```
1 subgoal (ID 9)

  H0 : ~ (forall x : D, ~ p x)
  H1 : ~ (exists x : D, p x)
  ============================
  forall x : D, ~ p x
```

$$\dfrac{\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \forall x.\, \neg p(x)}{\bot}\, E\neg}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\, I[H_0]{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.
```

```
1 subgoal (ID 12)

  H0 : ~ (forall x : D, ~ p x)
  H1 : ~ (exists x : D, p x)
  x : D
  H2 : p x
  ============================
  False
```

$$\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\bot}{\neg p(x)}\, I[H_2]\neg}{\forall x.\, \neg p(x)}\, I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}\, E\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\, I[H_0]{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.
apply H1.
```

```
1 subgoal (ID 13)

  H0 : ~ (forall x : D, ~ p x)
  H1 : ~ (exists x : D, p x)
  x : D
  H2 : p x
  ============================
  exists x0 : D, p x0
```

$$\dfrac{\dfrac{\dfrac{[\neg(\forall x.\,\neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\,p(x))^{H_1}] \qquad \exists x.\,p(x)}{\bot}\,E\!\to}{\neg p(x)}\,I[H_2]\neg}{\forall x.\,\neg p(x)}\,I\forall}{\bot}\,E\neg}{\neg\neg(\exists x.\,p(x))}\,I[H_1]\neg}{\neg(\forall x.\,\neg p(x)) \to \neg\neg(\exists x.\,p(x))}\,I[H_0]\!\to$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.
apply H1.
exists x.
```

```
1 subgoal (ID 15)

  H0 : ~ (forall x : D, ~ p x)
  H1 : ~ (exists x : D, p x)
  x : D
  H2 : p x
  ============================
  p x
```

$$\dfrac{\dfrac{[\neg(\forall x.\,\neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\,p(x))^{H_1}] \qquad \dfrac{p(x)}{\exists x.\,p(x)}\,I\exists}{\bot}\,E{\to}}{\neg p(x)}\,I[H_2]\neg}{\forall x.\,\neg p(x)}\,I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\,p(x))}\,I[H_1]\neg}\,E\neg}{\neg(\forall x.\,\neg p(x)) \to \neg\neg(\exists x.\,p(x))}\,I[H_0]{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.
apply H1.
exists x.
apply H2.
```

```
No more subgoals.
```

$$\dfrac{\dfrac{\dfrac{[\neg(\forall x.\, \neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\, p(x))^{H_1}] \qquad \dfrac{[p(x)^{H_2}]}{\exists x.\, p(x)}\, I\exists}{\bot}\, E{\to}}{\neg p(x)}\, I[H_2]\neg}{\forall x.\, \neg p(x)}\, I\forall}{\bot}\, E\neg}{\neg\neg(\exists x.\, p(x))}\, I[H_1]\neg}{\neg(\forall x.\, \neg p(x)) \to \neg\neg(\exists x.\, p(x))}\, I[H_0]{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.
apply H1.
exists x.
apply H2.
Qed.
```

$$\dfrac{\dfrac{\dfrac{[\neg(\forall x.\,\neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\,p(x))^{H_1}] \qquad \dfrac{[p(x)^{H_2}]}{\exists x.\,p(x)}\,I\exists}{\bot}\,E{\to}}{\neg p(x)}\,I[H_2]\neg}{\forall x.\,\neg p(x)}\,I\forall}{\bot}\,E\neg}{\neg\neg(\exists x.\,p(x))}\,I[H_1]\neg}{\neg(\forall x.\,\neg p(x)) \to \neg\neg(\exists x.\,p(x))}\,I[H_0]{\to}$$

the second example in Coq, from right to left

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.
apply H1.
exists x.
apply H2.
Qed.
```

$$
\dfrac{\dfrac{[\neg(\forall x.\,\neg p(x))^{H_0}] \qquad \dfrac{\dfrac{\dfrac{[\neg(\exists x.\,p(x))^{H_1}] \qquad \dfrac{[p(x)^{H_2}]}{\exists x.\,p(x)}\,I\exists}{\bot}\,E{\rightarrow}}{\neg p(x)}\,I[H_2]\neg}{\forall x.\,\neg p(x)}\,I\forall}{\dfrac{\bot}{\neg\neg(\exists x.\,p(x))}\,I[H_1]\neg}\,E\neg}{\neg(\forall x.\,\neg p(x)) \rightarrow \neg\neg(\exists x.\,p(x))}\,I[H_0]{\rightarrow}
$$

proof rules versus Coq tactics

$I{\rightarrow}\ \ I\forall$	`intro` `intros`
$E{\rightarrow}\ \ E\forall$	`apply`
$E\wedge\ \ E\vee\ \ E\exists$	`elim` `destruct` `intro`-patterns
$I\wedge$	`split`
$I\vee$	`left` `right`
$I\exists$	`exists`

detours

proof normalization

detour = introduction rule directly followed by a elimination rule for the same connective

detours

proof normalization

detour $=$ introduction rule directly followed by a elimination rule for the same connective

cut $=$ corresponding notion in sequent calculus

can be eliminated $\longrightarrow$ reduction of the proof term

proof normalization

detour $=$ introduction rule directly followed by a elimination rule for the same connective

cut $=$ corresponding notion in sequent calculus

can be eliminated $\longrightarrow$ reduction of the proof term

detour for implication:

'prove a lemma A and then prove B using this lemma'
detour elimination is inlining the proof$_1$ of the lemma everywhere

$$\dfrac{\dfrac{\begin{array}{c}[A^H]\\ \vdots_2\\ B\end{array}}{A \to B}\,I[H]{\to} \qquad \begin{array}{c}\vdots_1\\ A\end{array}}{B}\,E{\to} \qquad \longrightarrow_\beta \qquad \begin{array}{c}\vdots_1\\ A\\ \vdots_2\\ B\end{array}$$

detour for the universal quantifer

generalize the statement $A[x := M]$ to A with arbitrary x
elimination is specializing the proof$_1$ to M

$$\dfrac{\dfrac{\begin{matrix}\vdots_1 \\ A\end{matrix}}{\forall x.\, A}\, I\forall}{A[x := M]}\, E\forall \qquad \longrightarrow_\beta \qquad \begin{matrix}\vdots_1[x := M] \\ A[x := M]\end{matrix}$$

$$(\lambda x : D.\, H_1)M \to_\beta H_1[x := M]$$

STT versus λP syntax

STT:

$$A, B ::= a \mid A \to B$$
$$M, N ::= x \mid M\,N \mid \lambda x : A.\, M$$
$$\Gamma ::= \cdot \mid \Gamma,\ x : A$$
$$\mathcal{J} ::= \Gamma \vdash M : A$$

λP:

$$s ::= * \mid \Box$$
$$M, N, A, B ::= x \mid M\,N \mid \lambda x : A.\, M \mid \overbrace{\Pi x : A.\, B}^{A \to B} \mid s$$
$$\Gamma ::= \cdot \mid \Gamma,\ x : A$$
$$\mathcal{J} ::= \Gamma \vdash M : A$$

λP

STT versus λP syntax

STT:

$$
\begin{aligned}
A, B &::= a \mid A \to B \\
M, N &::= x \mid M\,N \mid \lambda x : A.\, M \\
\Gamma &::= \cdot \mid \Gamma,\, x : A \\
\mathcal{J} &::= \Gamma \vdash M : A
\end{aligned}
$$

λP:

$$
\begin{aligned}
s &::= * \mid \square \\
M, N, A, B &::= x \mid M\,N \mid \lambda x : A.\, M \mid \overbrace{\Pi x : A.\, B}^{A \to B} \mid s \\
\Gamma &::= \cdot \mid \Gamma,\, x : A \\
\mathcal{J} &::= \Gamma \vdash M : A
\end{aligned}
$$

$$
a : *,\ b : *,\ f : a \to b,\ x : a \vdash f\,x : b
$$

STT versus λP syntax

STT:

$$A, B ::= a \mid A \to B$$
$$M, N ::= x \mid M\,N \mid \lambda x : A.\, M$$
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$$s ::= * \mid \square$$
$$M, N, A, B ::= x \mid M\,N \mid \lambda x : A.\, M \mid \overbrace{\Pi x : A.\, B}^{A \to B} \mid s$$
$$\Gamma ::= \cdot \mid \Gamma,\ x : A$$
$$\mathcal{J} ::= \Gamma \vdash M : A$$

$$\overbrace{a : *,\ b : *,\ f : a \to b,\ x : a}^{\Gamma} \vdash \overbrace{f\,x}^{M} : \overbrace{b}^{A}$$

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A.\, M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A.\, M \mid \Pi x : A.\, M \mid * \mid \Box$$

STT

3 rules

$$x \mid MN \mid \lambda x : A.\, M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A.\, M \mid \Pi x : A.\, M \mid * \mid \Box$$

box does not have a type
$\longrightarrow$ no rule for box

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A.\, M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A.\, M \mid \Pi x : A.\, M \mid *$$

box does not have a type
$\longrightarrow$ no rule for box

variables have two rules

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A.\, M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A.\, M \mid \Pi x : A.\, M \mid *$$

box does not have a type
$\longrightarrow$ no rule for box

variables have two rules

conversion rule

typing rules

λP

$$\frac{}{\vdash * : \square}$$

typing rules

λP

$$\overline{\vdash * : \square}$$

axiom rule

start rule

λP

$$\frac{}{\vdash * : \square}$$

axiom rule

start rule

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A .\, B : s}$$

product rule

typing rules

STT	λP
	$$\frac{}{\vdash * : \square}$$ axiom rule start rule
	$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$ product rule
$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.\, M : A \to B}$$	abstraction rule

typing rules

STT	λP
	$$\frac{}{\vdash * : \Box}$$ axiom rule start rule
	$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$ product rule
$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.\, M : A \to B}$$	$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}$$ abstraction rule

typing rules

STT

λP

$$\frac{}{\vdash * : \Box}$$

axiom rule

start rule

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

product rule

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A.\, M : A \to B}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}$$

abstraction rule

typing rules (continued)

STT

$$\frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

λP

application rule

typing rules (continued)

STT

$$\frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

λP

$$\frac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

application rule

typing rules (continued)

STT	λP
$$\frac{\Gamma \vdash F : A \to B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$	$$\frac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$
	application rule

$$\begin{aligned} \mathsf{truesvec} &: \Pi n : \mathsf{nat}.\, \mathsf{vec}\ n \\ \mathsf{truesvec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) &: \mathsf{vec}\ (\mathsf{S}\ (\mathsf{S}\ \mathsf{O})) \end{aligned}$$

typing rules (continued)

STT

λP

$$\frac{}{\Gamma \vdash x : A}\ (x : A) \in \Gamma$$

variable rule

typing rules (continued)

STT

(context Γ is a set)

λP

(context Γ is a list)

correct: $a : *,\ x : a \vdash x : a$

incorrect: $x : a,\ a : * \vdash x : a$

$$\frac{}{\Gamma \vdash x : A}\ (x : A) \in \Gamma$$

variable rule

typing rules (continued)

STT

(context Γ is a set)

$$\frac{}{\Gamma \vdash x : A}\,(x : A) \in \Gamma$$

λP

(context Γ is a list)

correct: $a : *,\, x : a \vdash x : a$
incorrect: $x : a,\, a : * \vdash x : a$

$$\frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}$$

variable rule

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

weakening rule

the final typing rule

$$\mathsf{vecappend} : \Pi n_1 : \mathsf{nat}.\, \Pi n_2 : \mathsf{nat}.\\ \mathsf{vec}\ n_1 \to \mathsf{vec}\ n_2 \to \mathsf{vec}\ (\mathsf{add}\ n_1\ n_2)$$

$\mathsf{vecappend}\ 3\ 4\ \ v_1\ v_2 : \mathsf{vec}\ (\mathsf{add}\ 3\ 4)$

$\mathsf{vecappend}\ 3\ 4\ \ v_1\ v_2 : \mathsf{vec}\ 7$

the final typing rule

$$\mathsf{vecappend} : \Pi n_1 : \mathsf{nat}.\, \Pi n_2 : \mathsf{nat}.\ \mathsf{vec}\ n_1 \to \mathsf{vec}\ n_2 \to \mathsf{vec}\ (\mathsf{add}\ n_1\ n_2)$$

$\mathsf{vecappend}\ 3\ 4\ \ v_1\ v_2 : \mathsf{vec}\ (\mathsf{add}\ 3\ 4)$

$\mathsf{vecappend}\ 3\ 4\ \ v_1\ v_2 : \mathsf{vec}\ 7$

λP

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash A' : s}{\Gamma \vdash M : A'}\ A =_\beta A'$$

conversion rule

A and A' are convertible

$=_\beta$ is defined on preterms

λP on one slide

$$M, F, A, B ::= x \mid FM \mid \lambda x : A.\, M \mid \Pi x : A.\, B \mid s$$
$$s ::= * \mid \Box$$

$$\frac{}{\vdash * : \Box} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

$$\frac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash A' : s}{\Gamma \vdash M : A'}\ A =_\beta A'$$

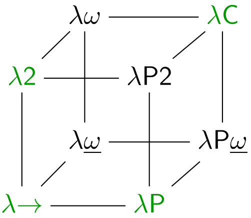
only the product rules differ:

$\lambda{\to}$ $$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x : A.\, B : *}$$

λP $$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

λ2 $$\frac{\Gamma \vdash A : s \qquad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x : A.\, B : *}$$

λC $$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_2}$$



$$\frac{}{\vdash s_1 : s_2}\,(s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_3}\,(s_1, s_2, s_3) \in \mathcal{R}$$

$$\frac{}{\vdash s_1 : s_2}\ (s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

	$\mathcal{S}$ sorts	$\mathcal{A}$ axioms	$\mathcal{R}$ rules
$\lambda{\to}$	$\{*, \Box\}$	$\{(*, \Box)\}$	$\{(*, *, *)\}$
λP	$\{*, \Box\}$	$\{(*, \Box)\}$	$\{(*, *, *), (*, \Box, \Box)\}$

$$\frac{}{\vdash s_1 : s_2}\ (s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

	$\mathcal{S}$ sorts	$\mathcal{A}$ axioms	$\mathcal{R}$ rules
$\lambda *$	$\{*\}$	$\{(*, *)\}$	$\{(*, *, *)\}$
$\lambda\!\to$	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *)\}$
λP	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square)\}$

$$\frac{}{\vdash s_1 : s_2}\ (s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

	$\mathcal{S}$ sorts	$\mathcal{A}$ axioms	$\mathcal{R}$ rules
$\lambda *$	$\{*\}$	$\{(*, *)\}$	$\{(*, *, *)\}$
$\lambda\!\rightarrow$	$\{*, \Box\}$	$\{(*, \Box)\}$	$\{(*, *, *)\}$
λP	$\{*, \Box\}$	$\{(*, \Box)\}$	$\{(*, *, *), (*, \Box, \Box)\}$
λC	$\{*, \Box\}$	$\{(*, \Box)\}$	$\{(*, *, *), (*, \Box, \Box), (\Box, *, *), (\Box, \Box, \Box)\}$

$$\frac{}{\vdash s_1 : s_2}\ (s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.\, B : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

	$\mathcal{S}$ sorts	$\mathcal{A}$ axioms	$\mathcal{R}$ rules
$\lambda *$	$\{*\}$	$\{(*, *)\}$	$\{(*, *, *)\}$
$\lambda\!\to$	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *)\}$
λP	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square)\}$
λC	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square),$ $(\square, *, *), (\square, \square, \square)\}$
λPRED	$\{*_s, \square_s, *_p, \square_p\}$	$\{(*_s, \square_s), (*_p, \square_p)\}$	$\{(*_s, *_s, *_s), (*_s, \square_s, \square_s),$ $(*_p, *_p, *_p), (*_s, *_p, *_p)\}$

example PTS type derivation

$$a : *, \, x : a \vdash (\lambda y : a.\, y)x : a$$

example PTS type derivation

$$a : *,\ x : a \vdash (\lambda y : a.\, y)x : a$$

$$\frac{}{\vdash * : \square} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\frac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \frac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\frac{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}$$

$$\frac{}{\vdash * : \Box} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\frac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \frac{\Gamma \vdash F : \Pi x : A.\, B \qquad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\dfrac{\dfrac{a:*,\, x:a,\, y:a \vdash y:a \qquad a:*,\, x:a \vdash a \to a : *}{a:*,\, x:a \vdash (\lambda y:a.\, y) : a \to a} \qquad a:*,\, x:a \vdash x:a}{a:*,\, x:a \vdash (\lambda y:a.\, y)x : a}$$

$$\dfrac{}{\vdash * : \square} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x:A \vdash x:A} \qquad \dfrac{\Gamma \vdash M:A \qquad \Gamma \vdash B:s}{\Gamma,\, y:B \vdash M:A}$$

$$\dfrac{\Gamma,\, x:A \vdash M:B \qquad \Gamma \vdash \Pi x:A.\, B : s}{\Gamma \vdash \lambda x:A.\, M : \Pi x:A.\, B} \qquad \dfrac{\Gamma \vdash F : \Pi x:A.\, B \qquad \Gamma \vdash M:A}{\Gamma \vdash FM : B[x:=M]}$$

example PTS type derivation

$$\dfrac{\dfrac{\dfrac{a : *,\, x : a \vdash a : *}{a : *,\, x : a,\, y : a \vdash y : a} \quad a : *,\, x : a \vdash a \to a : *}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a} \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}$$

$$\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{a : * \vdash a : * \quad a : * \vdash a : *}{a : *,\, x : a \vdash a : *}}{a : *,\, x : a,\, y : a \vdash y : a} \quad a : *,\, x : a \vdash a \to a : *}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a} \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}$$

$$\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash * : \square}{a : * \vdash a : *} \quad a : * \vdash a : *}{a : *,\, x : a \vdash a : *}}{a : *,\, x : a,\, y : a \vdash y : a} \quad a : *,\, x : a \vdash a \to a : *}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a} \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y) x : a}$$

$$\dfrac{}{\vdash * : \square} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \quad a : * \vdash a : *}{a : *,\, x : a \vdash a : *}}{a : *,\, x : a,\, y : a \vdash y : a} \quad a : *,\, x : a \vdash a \to a : *}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a} \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}$$

$$\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \quad \dfrac{\vdash * : \Box}{a : * \vdash a : *}}{a : *,\, x : a \vdash a : *}}{a : *,\, x : a,\, y : a \vdash y : a} \quad a : *,\, x : a \vdash a \to a : *}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a} \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}
$$

$$
\frac{}{\vdash * : \Box} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\frac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B} \qquad \frac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
$$

example PTS type derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
      \quad
      \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
    }{a : *,\, x : a \vdash a : *}
  }{a : *,\, x : a,\, y : a \vdash y : a}
  \quad
  a : *,\, x : a \vdash a \to a : *
}{
  \dfrac{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}
}
$$

$$
\dfrac{}{\vdash * : \square}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}
\qquad
\dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
$$

example PTS type derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\vdash * : \Box}
      }{a : * \vdash a : *}
      \qquad
      \dfrac{
        \dfrac{}{\vdash * : \Box}
      }{a : * \vdash a : *}
    }{
      \dfrac{a : *,\, x : a \vdash a : *}{a : *,\, x : a,\, y : a \vdash y : a}
    }
    \qquad
    \dfrac{\begin{array}{c}\text{(next slide)}\\ \vdots\end{array}}{a : *,\, x : a \vdash a \to a : *}
  }{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a}
  \qquad
  \color{red}{a : *,\, x : a \vdash x : a}
}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}
$$

$$
\dfrac{}{\vdash * : \Box}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}
\qquad
\dfrac{\Gamma \vdash F : \Pi x : A.\, B \qquad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
$$

example PTS type derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
      \qquad
      \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
    }{a : *,\, x : a \vdash a : *}
  }{a : *,\, x : a,\, y : a \vdash y : a}
  \qquad
  \dfrac{\begin{matrix}\text{(next slide)}\\ \vdots\end{matrix}}{a : *,\, x : a \vdash a \to a : *}
}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a}
\qquad
\dfrac{a : * \vdash a : *}{a : *,\, x : a \vdash x : a}
$$

$$
\dfrac{}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}
$$

$$
\dfrac{}{\vdash * : \Box}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}
\qquad
\dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
$$

example PTS type derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
      \qquad
      \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
    }{a : *,\, x : a \vdash a : *}
  }{a : *,\, x : a,\, y : a \vdash y : a}
  \qquad
  \dfrac{\substack{\text{(next slide)} \\ \vdots}}{a : *,\, x : a \vdash a \to a : *}
}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a}
\qquad
\dfrac{\dfrac{\vdash * : \Box}{a : * \vdash a : *}}{a : *,\, x : a \vdash x : a}
\over
a : *,\, x : a \vdash (\lambda y : a.\, y)x : a
$$

$$
\dfrac{}{\vdash * : \Box}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma,\, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}
\qquad
\dfrac{\Gamma \vdash F : \Pi x : A.\, B \qquad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
$$

example PTS type derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
      \qquad
      \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
    }{a : *,\, x : a \vdash a : *}
  }{a : *,\, x : a,\, y : a \vdash y : a}
  \qquad
  \dfrac{\begin{matrix}\text{(next slide)}\\ \vdots\end{matrix}}{a : *,\, x : a \vdash a \to a : *}
}{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a}
\qquad
\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}}{a : *,\, x : a \vdash x : a}
$$

$$
\dfrac{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a \qquad a : *,\, x : a \vdash x : a}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}
$$

$$
\dfrac{}{\vdash * : \Box}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma,\, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A.\, B : s}{\Gamma \vdash \lambda x : A.\, M : \Pi x : A.\, B}
\qquad
\dfrac{\Gamma \vdash F : \Pi x : A.\, B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
$$

example PTS type derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
        \quad
        \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}
      }{a : *,\, x : a \vdash a : *}
    }{a : *,\, x : a,\, y : a \vdash y : a}
    \quad
    \dfrac{\begin{matrix}\text{(next slide)}\\ \vdots\end{matrix}}{a : *,\, x : a \vdash a \to a : *}
  }{a : *,\, x : a \vdash (\lambda y : a.\, y) : a \to a}
  \quad
  \dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}}{a : *,\, x : a \vdash x : a}
}{a : *,\, x : a \vdash (\lambda y : a.\, y)x : a}
$$

example PTS type derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \quad \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}}{a : *,\ x : a \vdash a : *}}{a : *,\ x : a,\ y : a \vdash y : a} \quad \dfrac{\begin{array}{c}\text{(next slide)}\\ \vdots\end{array}}{a : *,\ x : a \vdash a \to a : *}}{a : *,\ x : a \vdash (\lambda y : a.\, y) : a \to a} \quad \dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}}{a : *,\ x : a \vdash x : a}}{a : *,\ x : a \vdash (\lambda y : a.\, y)x : a}$$

$$\dfrac{\dfrac{\dfrac{}{x : a,\ y : a \vdash y : a}}{x : a \vdash (\lambda y : a.\, y) : a \to a} \quad \dfrac{}{x : a \vdash x : a}}{x : a \vdash (\lambda y : a.\, y)x : a}$$

example PTS type derivation

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
\quad
\dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
}{a : *,\ x : a \vdash a : *}
}{a : *,\ x : a,\ y : a \vdash y : a}
\quad
\dfrac{\begin{array}{c}\text{(next slide)}\\ \vdots\end{array}}{\color{red}{a : *,\ x : a \vdash a \to a : *}}
}{a : *,\ x : a \vdash (\lambda y : a.\, y) : a \to a}
\quad
\dfrac{\dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}}{a : *,\ x : a \vdash x : a}
}{a : *,\ x : a \vdash (\lambda y : a.\, y)x : a}
$$

$$
\dfrac{
\dfrac{\dfrac{}{x : a,\ y : a \vdash y : a}}{x : a \vdash (\lambda y : a.\, y) : a \to a}
\quad
\dfrac{}{x : a \vdash x : a}
}{x : a \vdash (\lambda y : a.\, y)x : a}
$$

example PTS type derivation (continued)

$$a : *,\, x : a \vdash a \to a : *$$

$$\frac{}{\vdash * : \square} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

$$a : *,\ x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *$$

$$\frac{}{\vdash * : \square} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\ x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\ y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma,\ x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

$$\frac{a : * \vdash (\Pi y : a.\, a) : * \qquad a : * \vdash a : *}{a : *,\ x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}$$

$$\frac{}{\vdash * : \Box} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\ x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\ y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma,\ x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

example PTS type derivation (continued)

$$\dfrac{\dfrac{a : * \vdash a : * \qquad a : *,\, y : a \vdash a : *}{a : * \vdash (\Pi y : a.\, a) : *} \qquad a : * \vdash a : *}{a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}$$

$$\dfrac{}{\vdash * : \square} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\dfrac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

example PTS type derivation (continued)

$$
\dfrac{\dfrac{\dfrac{\vdash * : \Box}{a : * \vdash a : *} \qquad a : *, y : a \vdash a : *}{a : * \vdash (\Pi y : a.\, a) : *} \qquad a : * \vdash a : *}{a : *, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}
$$

$$
\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

example PTS type derivation (continued)

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \qquad a : *,\, y : a \vdash a : *}{a : * \vdash (\Pi y : a.\, a) : *} \qquad a : * \vdash a : *}{a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}
$$

$$
\frac{}{\vdash * : \Box} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\frac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

example PTS type derivation (continued)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \qquad \dfrac{a : * \vdash a : * \quad a : * \vdash a : *}{a : *,\ y : a \vdash a : *}}{a : * \vdash (\Pi y : a.\, a) : *} \qquad a : * \vdash a : *}{a : *,\ x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}$$

$$\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\ x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\ y : B \vdash M : A}$$

$$\dfrac{\Gamma \vdash A : * \quad \Gamma,\ x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

example PTS type derivation (continued)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \qquad \dfrac{\dfrac{\vdash * : \Box}{a : * \vdash a : *} \quad a : * \vdash a : *}{a : *,\, y : a \vdash a : *}}{a : * \vdash (\Pi y : a.\, a) : *} \qquad a : * \vdash a : *}{a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}
$$

$$
\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

example PTS type derivation (continued)

$$
\dfrac{
  \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
  \qquad
  \dfrac{
    \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
    \quad
    a : * \vdash a : *
  }{a : *,\, y : a \vdash a : *}
}{
  \dfrac{a : * \vdash (\Pi y : a.\, a) : * \qquad\qquad a : * \vdash a : *}{a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}
}
$$

$$
\dfrac{}{\vdash * : \square}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

example PTS type derivation (continued)

$$\frac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \qquad \dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \quad \dfrac{\vdash * : \Box}{a : * \vdash a : *}}{a : *,\, y : a \vdash a : *}}{\dfrac{a : * \vdash (\Pi y : a.\, a) : * \qquad\qquad a : * \vdash a : *}{a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}}$$

$$\frac{}{\vdash * : \Box} \qquad \frac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

example PTS type derivation (continued)

$$
\dfrac{
  \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
  \qquad
  \dfrac{
    \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}
    \quad
    \dfrac{\dfrac{}{\vdash {\color{green} * : \square}}}{a : * \vdash a : *}
  }{a : *,\, y : a \vdash a : *}
}{a : * \vdash (\Pi y : a.\, a) : *}
\qquad
{\color{red} a : * \vdash a : *}
$$

$$
\overline{\quad a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : * \quad}
$$

$$
\dfrac{}{\vdash {\color{green} * : \square}}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

example PTS type derivation (continued)

$$\dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \qquad \dfrac{\dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *} \quad \dfrac{\dfrac{}{\vdash * : \Box}}{a : * \vdash a : *}}{a : *,\, y : a \vdash a : *}}{a : * \vdash (\Pi y : a.\, a) : *} \qquad \dfrac{\vdash * : \Box}{a : * \vdash a : *}$$

$$\dfrac{a : * \vdash (\Pi y : a.\, a) : * \qquad a : * \vdash a : *}{a : *,\, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}$$

$$\dfrac{}{\vdash * : \Box} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma,\, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma,\, y : B \vdash M : A}$$

$$\dfrac{\Gamma \vdash A : * \quad \Gamma,\, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}$$

example PTS type derivation (continued)

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{\vdash \ast : \Box}}{a : \ast \vdash a : \ast}
    \qquad
    \dfrac{
      \dfrac{\dfrac{}{\vdash \ast : \Box}}{a : \ast \vdash a : \ast}
      \quad
      \dfrac{\dfrac{}{\vdash \ast : \Box}}{a : \ast \vdash a : \ast}
    }{a : \ast, y : a \vdash a : \ast}
  }{a : \ast \vdash (\Pi y : a.\, a) : \ast}
  \qquad
  \dfrac{\dfrac{}{\vdash \ast : \Box}}{a : \ast \vdash a : \ast}
}{a : \ast, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : \ast}
$$

$$
\dfrac{}{\vdash \ast : \Box}
\qquad
\dfrac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}
\qquad
\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma \vdash A : \ast \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

example PTS type derivation (continued)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *} \qquad \dfrac{\dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *} \quad \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}}{a : *, y : a \vdash a : *}}{a : * \vdash (\Pi y : a.\, a) : *} \qquad \dfrac{\dfrac{}{\vdash * : \square}}{a : * \vdash a : *}}{a : *, x : a \vdash \underbrace{a \to a}_{\Pi y : a.\, a} : *}
$$

$$
\dfrac{}{\vdash * : \square} \qquad \dfrac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \qquad \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}
$$

$$
\dfrac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A.\, B : s}
$$

computing the sort of a type

$$
\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
\mathsf{type}_\Gamma(\Pi x : A.\, B) &= \mathsf{type}_{\Gamma, x:A}(B) \\
\mathsf{type}_\Gamma(A \to B) &= \mathsf{type}_\Gamma(B)
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{type}_\Gamma(*) &= \Box \\
\mathsf{type}_\Gamma(x) &= \Gamma(x)
\end{aligned}
$$

$$
\mathsf{type}_\Gamma(F M) = \ldots
$$

summary

- predicate logic
- dependent types
  - vectors
  - Curry-Howard
- λP
  - lambda cube
  - PTSs
- detours

contents