second order logic & polymorphism

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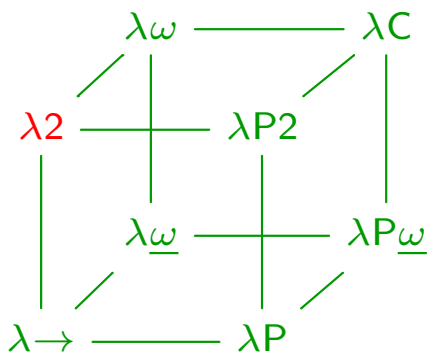
Type Theory & Coq
2024–2025
Radboud University Nijmegen

October 11, 2024



introduction

the two remaining chapters

propositional logic	$\lambda\!\!\rightarrow$	simple types
predicate logic	λP	dependent types
second order logic	$\lambda 2$	polymorphic types
the Coq logic	CIC	inductive types

recap: dependent types

$$(\lambda x : A.\, M) : A \to B$$

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$$(\lambda x : A.\, M) : A \to B$$
$$(\lambda x : A.\, M) : (\Pi x : A.\, B)$$

recap: dependent types

$$(\lambda x : A.\, M) : A \to B$$
$$(\lambda x : A.\, M) : (\Pi x : A.\, B)$$

```
(fun x : A => M) : A -> B
(fun x : A => M) : (forall x : A, B)
```

recap: inductive types

```
Inductive nat : Set :=
| O : nat
| S : nat -> nat.

Fixpoint add (n m : nat) : nat :=
  match n with
  | O => m
  | S n' => S (add n' m)
  end.

Lemma add_n_O (n : nat) :
  add n O = n.
induction n as [|n' IH].
- reflexivity.
- simpl. rewrite IH.
  reflexivity.
Qed.
```

recap: inductive types

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Inductive nat : Set :=
| O : nat
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induction n as [|n' IH].
- reflexivity.
- simpl. rewrite IH.
  reflexivity.
Qed.
```

$E[\Gamma] \vdash$ `nat_ind` : ...

today

- second order propositional logic
- polymorphism

- inversion
- some tactics
- program extraction

- λP as a logical framework
- a classical paradox in $\lambda 2$

second order propositional logic

higher order logic

logic:	quantifies over:
first order predicate logic	objects
second order predicate logic	objects sets of objects
third order predicate logic	objects sets of objects sets of sets of objects
	etcetera

second order propositional logic

higher order logic

logic:	quantifies over:
first order predicate logic	objects
second order predicate logic	objects predicates
third order predicate logic	objects predicates predicates of predicates
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second order propositional logic

higher order logic

logic:	quantifies over:
first order predicate logic	objects
second order predicate logic	objects predicates
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first order propositional logic	~~objects~~
second order propositional logic	~~objects~~ ~~predicates~~ propositions

second order propositional logic

higher order logic

logic:	quantifies over:
first order predicate logic	objects
second order predicate logic	objects predicates
third order predicate logic	objects predicates predicates of predicates
first order propositional logic	~~objects~~
second order propositional logic	~~objects~~ ~~predicates~~ propositions

equivalent: quantification over $\{\text{sets}, \text{functions}, \text{predicates}\}$

minimal second order propositional logic

syntax

$$A, B ::= a \mid A \to B \mid \forall a.\, A$$

rules

$$\frac{\begin{array}{c}[A^x] \\ \vdots \\ B\end{array}}{A \to B}\, I[x]\to \qquad \frac{\begin{array}{cc}\vdots & \vdots \\ A \to B & A\end{array}}{B}\, E\to$$

$$\frac{\begin{array}{c}\vdots \\ A\end{array}}{\forall a.\, A}\, I\forall \qquad \frac{\begin{array}{c}\vdots \\ \forall a.\, A\end{array}}{A[a := B]}\, E\forall$$

variable condition of $I\forall$: a not free in available assumptions

example proof

$$\forall a.\, (\forall b.\, b) \to a$$

example proof

$$\frac{(\forall b.\, b) \to a}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

example proof

$[\forall b.\, b^x]$

$$\dfrac{\dfrac{\overset{a}{}}{(\forall b.\, b) \to a}\, I[x]{\to}}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

example proof

$[\forall b.\, b^x]$

$$\dfrac{\dfrac{\dfrac{[\forall b.\, b^x]}{a}\, E\forall}{(\forall b.\, b) \to a}\, I[x]\to}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

example proof

$$\dfrac{\dfrac{\dfrac{[\forall b.\, b^x]}{a}\, E\forall}{(\forall b.\, b) \to a}\, I[x]\to}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

$$\lambda a : *.\, \lambda x : (\Pi b : *.\, b).\, xa$$

example proof

$$\dfrac{\dfrac{\dfrac{[\forall b.\, b^x]}{a}\, E\forall}{(\forall b.\, b) \to a}\, I[x]\to}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

$$\begin{array}{c} \lambda a : *.\, \lambda x : (\Pi b : *.\, b).\, xa \\ \vdots \\ \Pi a : *.\, (\Pi b : *.\, b) \to a \end{array}$$

example proof

$$\dfrac{\dfrac{\dfrac{[\forall b.\, b^x]}{a}\, E\forall}{(\forall b.\, b) \to a}\, I[x]\to}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

$$\lambda a : *.\, \lambda x : (\Pi b : *.\, b).\, xa$$
$$\vdots$$
$$\Pi a : *.\, (\Pi b : *.\, b) \to a$$

```
fun (a : Prop) (x : (forall b : Prop, b)) => x a
```
$$\vdots$$
```
forall a : Prop, (forall b : Prop, b) -> a
```

example proof

```
Lemma seven
    (a : Prop) :
  (forall b : Prop,
    b) -> a.
intros x.
apply x.
Qed.
```

$$\dfrac{\dfrac{\dfrac{[\forall b.\, b^x]}{a}\, E\forall}{(\forall b.\, b) \to a}\, I[x]\to}{\forall a.\, (\forall b.\, b) \to a}\, I\forall$$

$$\lambda a : *.\, \lambda x : (\Pi b : *.\, b).\, x a$$
$$\vdots$$
$$\Pi a : *.\, (\Pi b : *.\, b) \to a$$

```
fun (a : Prop) (x : (forall b : Prop, b)) => x a
```
$$\vdots$$
```
forall a : Prop, (forall b : Prop, b) -> a
```

constructive second order propositional logic

syntax

$$A, B ::= a \mid A \to B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \bot \mid \forall a.\, A \mid \exists a.\, A$$

rules

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{\forall a.\, A}\, I\forall \qquad \frac{\begin{array}{c}\vdots\\ \forall a.\, A\end{array}}{A[a := B]}\, E\forall$$

$$\frac{\begin{array}{c}\vdots\\ A[a := B]\end{array}}{\exists a.\, A}\, I\exists \qquad \frac{\begin{array}{c}\vdots\\ \exists a.\, A\end{array} \quad \begin{array}{c}\vdots\\ \forall a.\, A \to C\end{array}}{C}\, E\exists$$

variable condition of $E\exists$: a not free in C

example proof with existential quantifier

$$\forall a.\, (\exists b.\, a) \leftrightarrow a$$

example proof with existential quantifier

$$\frac{(\exists b.\, a) \leftrightarrow a}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$$\dfrac{\dfrac{(\exists b.\, a) \to a \qquad\qquad a \to (\exists b.\, a)}{(\exists b.\, a) \leftrightarrow a}\, I\wedge}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$[\exists b.\, a^x]$

$$\dfrac{\dfrac{\dfrac{a}{(\exists b.\, a) \to a}\, I[x]\to \qquad a \to (\exists b.\, a)}{(\exists b.\, a) \leftrightarrow a}\, I\wedge}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$[\exists b.\, a^x]$

$$\dfrac{\dfrac{\dfrac{[\exists b.\, a^x] \quad \forall b.\, a \to a}{a}\, E\exists}{(\exists b.\, a) \to a}\, I[x]\to \qquad a \to (\exists b.\, a)}{(\exists b.\, a) \leftrightarrow a}\, I\wedge$$

$$\dfrac{(\exists b.\, a) \leftrightarrow a}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$[\exists b.\, a^x]$

$$\dfrac{\dfrac{\dfrac{[\exists b.\, a^x] \quad \forall b.\, a \to a}{a}\, E\exists}{(\exists b.\, a) \to a}\, I[x]\to \qquad a \to (\exists b.\, a)}{(\exists b.\, a) \leftrightarrow a}\, I\wedge$$

$$\dfrac{(\exists b.\, a) \leftrightarrow a}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$[\exists b.\, a^x]$

$$\dfrac{\dfrac{\dfrac{[\exists b.\, a^x] \quad \dfrac{a \to a}{\forall b.\, a \to a}\, I\forall}{a}\, E\exists}{(\exists b.\, a) \to a}\, I[x]\to \qquad a \to (\exists b.\, a)}{\dfrac{(\exists b.\, a) \leftrightarrow a}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall}\, I\wedge$$

example proof with existential quantifier

$[\exists b.\, a^x]$ $[a^y]$

$$\dfrac{\dfrac{\dfrac{[\exists b.\, a^x] \quad \dfrac{\dfrac{[a^y]}{a \to a}\, I[y]\to}{\forall b.\, a \to a}\, I\forall}{a}\, E\exists}{(\exists b.\, a) \to a}\, I[x]\to \qquad a \to (\exists b.\, a)}{(\exists b.\, a) \leftrightarrow a}\, I\wedge$$

$$\dfrac{(\exists b.\, a) \leftrightarrow a}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$[\exists b.\, a^x]$ $[a^y]$

$$\dfrac{\dfrac{[\exists b.\, a^x] \quad \dfrac{\dfrac{[a^y]}{a \to a}\, I[y]\!\to}{\forall b.\, a \to a}\, I\forall}{a}\, E\exists}{\dfrac{(\exists b.\, a) \to a}{(\exists b.\, a) \leftrightarrow a}\, I[x]\!\to} \qquad \dfrac{}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall \qquad a \to (\exists b.\, a)\, I\wedge$$

example proof with existential quantifier

$[\exists b.\, a^x]$ $[a^y]$ $[a^z]$

$$\dfrac{\dfrac{[\exists b.\, a^x] \quad \dfrac{\dfrac{[a^y]}{a \to a}\, I[y]\!\to}{\forall b.\, a \to a}\, I\forall}{a}\, E\exists}{\dfrac{\dfrac{a}{(\exists b.\, a) \to a}\, I[x]\!\to \quad \dfrac{\exists b.\, a}{a \to (\exists b.\, a)}\, I[z]\!\to}{(\exists b.\, a) \leftrightarrow a}\, I\wedge}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall$$

example proof with existential quantifier

$[\exists b.\, a^x]$ $[a^y]$ $[a^z]$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\exists b.\, a^x] \quad \dfrac{\dfrac{[a^y]}{a \to a} I[y]\to}{\forall b.\, a \to a} I\forall}{a} E\exists}{(\exists b.\, a) \to a} I[x]\to \quad \dfrac{\dfrac{[a^z]}{\exists b.\, a} I\exists}{a \to (\exists b.\, a)} I[z]\to}{(\exists b.\, a) \leftrightarrow a} I\wedge}{\forall a.\, (\exists b.\, a) \leftrightarrow a} I\forall$$

example proof with existential quantifier

$$\dfrac{\dfrac{[\exists b.\, a^x] \quad \dfrac{\dfrac{[a^y]}{a \to a}\, I[y]\to}{\forall b.\, a \to a}\, I\forall}{a}\, E\exists \quad \dfrac{\dfrac{[a^z]}{\exists b.\, a}\, I\exists}{a \to (\exists b.\, a)}\, I[z]\to}{\dfrac{(\exists b.\, a) \leftrightarrow a}{\forall a.\, (\exists b.\, a) \leftrightarrow a}\, I\forall}\, I\wedge \; I[x]\to$$

```
Lemma eight (a : Prop) :
  (exists b : Prop, a) <-> a.
split.
- intros [b y]. apply y.
- intros z. exists True. apply z.
Qed.
```

defining the logical operators in minimal logic

$$\bot := \forall c.\, c$$

$$\top := \forall c.\, c \to c$$

$$\neg A := \forall c.\, A \to c$$

$$A \wedge B := \forall c.\, (A \to B \to c) \to c$$

$$A \vee B := \forall c.\, (A \to c) \to (B \to c) \to c$$

$$\exists a.\, A := \forall c.\, (\forall a.\, A \to c) \to c$$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{A}\, El\wedge$$

$$A$$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\dfrac{\dfrac{\vdots}{A \wedge B}}{A}\, El\wedge \qquad \dfrac{\dfrac{\vdots}{\forall c.\, (A \to B \to c) \to c}}{(A \to B \to A) \to A}\, E\forall \qquad A \to B \to A}{A}\, E\to$$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\dfrac{\dfrac{\vdots}{A \wedge B}}{A}\, El\wedge \qquad \dfrac{\dfrac{\dfrac{\vdots}{\forall c.\, (A \to B \to c) \to c}}{(A \to B \to A) \to A}\, E\forall \qquad \dfrac{B \to A}{A \to B \to A}\, I[x]\to}{A}\, E\to$$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\dfrac{\dfrac{\vdots}{A \wedge B}\, El\wedge}{A}}{}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\forall c.\, (A \to B \to c) \to c}\, E\forall}{(A \to B \to A) \to A} \qquad \dfrac{\dfrac{[A^x]}{B \to A}\, I[y]\to}{A \to B \to A}\, I[x]\to}{A}\, E\to$$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\dfrac{\dfrac{\vdots}{A \wedge B}\, El\wedge}{A}}{}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\forall c.\, (A \to B \to c) \to c}\, E\forall}{(A \to B \to A) \to A} \quad \dfrac{\dfrac{[A^x]}{B \to A}\, I[y]\to}{A \to B \to A}\, I[x]\to}{A}\, E\to$$

$M : \forall c.\, (A \to B \to c) \to c$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\dfrac{\dfrac{\vdots}{A \wedge B}\, El\wedge}{A}}{}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\forall c.\, (A \to B \to c) \to c}\, E\forall}{(A \to B \to A) \to A} \quad \dfrac{\dfrac{[A^x]}{B \to A}\, I[y]\to}{A \to B \to A}\, I[x]\to}{A}\, E\to$$

$M : \forall c.\, (A \to B \to c) \to c$

$M\,A : (A \to B \to A) \to A$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\dfrac{\vdots}{A \wedge B}}{A}\, El\wedge$$

$$\dfrac{\dfrac{\vdots}{\forall c.\, (A \to B \to c) \to c}}{(A \to B \to A) \to A}\, E\forall \qquad \dfrac{\dfrac{[A^x]}{B \to A}\, I[y]\to}{A \to B \to A}\, I[x]\to$$
$$\dfrac{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}{A}\, E\to$$

$$M : \forall c.\, (A \to B \to c) \to c$$
$$M\, A : (A \to B \to A) \to A$$
$$(\lambda x : A.\, \lambda y : B.\, x) : A \to B \to A$$

admissibility of the proof rules

$A \wedge B := \forall c.\, (A \to B \to c) \to c$

$$\dfrac{\begin{array}{c}\vdots\\ A \wedge B\end{array}}{A}\, El\wedge$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall c.\, (A \to B \to c) \to c\end{array}}{(A \to B \to A) \to A}\, E\forall \qquad \dfrac{\dfrac{[A^x]}{B \to A}\, I[y]\to}{A \to B \to A}\, I[x]\to}{A}\, E\to$$

$$M : \forall c.\, (A \to B \to c) \to c$$

$$M\,A : (A \to B \to A) \to A$$

$$(\lambda x : A.\, \lambda y : B.\, x) : A \to B \to A$$

$$M\,A\,(\lambda x : A.\, \lambda y : B.\, x) : A$$

polymorphism

the polymorphic identity

$$(\lambda x : \mathsf{nat}.\, x) : \mathsf{nat} \to \mathsf{nat}$$

$$(\lambda x : \mathsf{bool}.\, x) : \mathsf{bool} \to \mathsf{bool}$$

$$\lambda a : *.\, (\lambda x : a.\, x) : \Pi a : *.\, a \to a$$

polymorphism

the polymorphic identity

$$(\lambda x : \mathsf{nat}.\, x) : \mathsf{nat} \to \mathsf{nat}$$

$$(\lambda x : \mathsf{bool}.\, x) : \mathsf{bool} \to \mathsf{bool}$$

$$\lambda a : *.\, (\lambda x : a.\, x) : \Pi a : *.\, a \to a$$

$$\mathsf{id} := \lambda a : *.\, \lambda x : a.\, x$$

$$\mathsf{id\ nat} \to_\beta \lambda x : \mathsf{nat}.\, x$$

$$\mathsf{id\ bool} \to_\beta \lambda x : \mathsf{bool}.\, x$$

polymorphism

the polymorphic identity

$$(\lambda x : \mathsf{nat}.\, x) : \mathsf{nat} \to \mathsf{nat}$$
$$(\lambda x : \mathsf{bool}.\, x) : \mathsf{bool} \to \mathsf{bool}$$
$$\lambda a : *.\, (\lambda x : a.\, x) : \Pi a : *.\, a \to a$$

$$\mathsf{id} := \lambda a : *.\, \lambda x : a.\, x$$
$$\mathsf{id\ nat} \to_\beta \lambda x : \mathsf{nat}.\, x$$
$$\mathsf{id\ bool} \to_\beta \lambda x : \mathsf{bool}.\, x$$

$$\Lambda a.\, M := \lambda a : *.\, M$$
$$\forall a.\, A := \Pi a : *.\, A$$

$$\Lambda a.\, \lambda x : a.\, x : \forall a.\, a \to a$$

stratified syntax

PTSs:

$$M, N, A, B := x \mid MN \mid \lambda x : A.\, M \mid \Pi x : A.\, B \mid * \mid \Box$$

$\lambda\!\rightarrow$:

$$A, B := a \mid A \rightarrow B$$
$$M, N := x \mid MN \mid \lambda x : A.\, M$$

$\lambda 2$:

$$A, B := a \mid A \rightarrow B \mid \forall a.\, A$$
$$M, N := x \mid MN \mid \lambda x : A.\, M \mid MA \mid \Lambda a.\, M$$

computing the type of a term

$$
\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
\mathsf{type}_\Gamma(\Pi x : A.\, B) &= \mathsf{type}_{\Gamma, x:A}(B) \\
\mathsf{type}_\Gamma(A \to B) &= \mathsf{type}_\Gamma(B) \\
\mathsf{type}_\Gamma(*) &= \Box \\
\mathsf{type}_\Gamma(x) &= \Gamma(x) \\
\mathsf{type}_\Gamma(FM) &= A[x := M] \quad \text{if } \mathsf{type}_\Gamma(F) =_\beta \Pi x : \mathsf{type}_\Gamma(M).\, A
\end{aligned}
$$

computing the type of a term

$$\mathsf{type}_\Gamma(\lambda x : A.\, M) = \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M)$$

$$\mathsf{type}_\Gamma(\Pi x : A.\, B) = \mathsf{type}_{\Gamma, x:A}(B)$$

$$\mathsf{type}_\Gamma(A \to B) = \mathsf{type}_\Gamma(B)$$

$$\mathsf{type}_\Gamma(*) = \Box$$

$$\mathsf{type}_\Gamma(x) = \Gamma(x)$$

$$\mathsf{type}_\Gamma(FM) = A[x := M] \quad \text{if } \mathsf{type}_\Gamma(F) =_\beta \Pi x : \mathsf{type}_\Gamma(M).\, A$$

$$\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x)$$

computing the type of a term

$$\mathsf{type}_\Gamma(\lambda x : A.\, M) = \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M)$$

$$\mathsf{type}_\Gamma(\Pi x : A.\, B) = \mathsf{type}_{\Gamma, x:A}(B)$$

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$$\mathsf{type}_\Gamma(FM) = A[x := M] \quad \text{if } \mathsf{type}_\Gamma(F) =_\beta \Pi x : \mathsf{type}_\Gamma(M).\, A$$

$$\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) = \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x)$$

computing the type of a term

$$\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
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\end{aligned}$$

$$\begin{aligned}
\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x)
\end{aligned}$$

computing the type of a term

$$\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
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&= \Pi a : *.\, \Pi x : a.\, a
\end{aligned}$$

computing the type of a term

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\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
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\end{aligned}$$

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\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x) \\
&= \Pi a : *.\, \Pi x : a.\, a \\
&= \Pi a : *.\, a \to a
\end{aligned}$$

computing the type of a term

$$\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
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\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x) \\
&= \Pi a : *.\, \Pi x : a.\, a \\
&= \Pi a : *.\, a \to a
\end{aligned}$$

$$\mathsf{type}(\Pi a : *.\, a \to a)$$

computing the type of a term

$$\mathsf{type}_\Gamma(\lambda x : A.\, M) = \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M)$$

$$\mathsf{type}_\Gamma(\Pi x : A.\, B) = \mathsf{type}_{\Gamma, x:A}(B)$$

$$\mathsf{type}_\Gamma(A \to B) = \mathsf{type}_\Gamma(B)$$

$$\mathsf{type}_\Gamma(*) = \Box$$

$$\mathsf{type}_\Gamma(x) = \Gamma(x)$$

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$$\begin{aligned}
\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x) \\
&= \Pi a : *.\, \Pi x : a.\, a \\
&= \Pi a : *.\, a \to a
\end{aligned}$$

$$\mathsf{type}(\Pi a : *.\, a \to a) = \mathsf{type}_{a:*}(a \to a)$$

computing the type of a term

$$\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
\mathsf{type}_\Gamma(\Pi x : A.\, B) &= \mathsf{type}_{\Gamma, x:A}(B) \\
\mathsf{type}_\Gamma(A \to B) &= \mathsf{type}_\Gamma(B) \\
\mathsf{type}_\Gamma(*) &= \Box \\
\mathsf{type}_\Gamma(x) &= \Gamma(x) \\
\mathsf{type}_\Gamma(FM) &= A[x := M] \quad \text{if } \mathsf{type}_\Gamma(F) =_\beta \Pi x : \mathsf{type}_\Gamma(M).\, A
\end{aligned}$$

$$\begin{aligned}
\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x) \\
&= \Pi a : *.\, \Pi x : a.\, a \\
&= \Pi a : *.\, a \to a
\end{aligned}$$

$$\begin{aligned}
\mathsf{type}(\Pi a : *.\, a \to a) &= \mathsf{type}_{a:*}(a \to a) \\
&= \mathsf{type}_{a:*}(a)
\end{aligned}$$

computing the type of a term

$$\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
\mathsf{type}_\Gamma(\Pi x : A.\, B) &= \mathsf{type}_{\Gamma, x:A}(B) \\
\mathsf{type}_\Gamma(A \to B) &= \mathsf{type}_\Gamma(B) \\
\mathsf{type}_\Gamma(*) &= \Box \\
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\mathsf{type}_\Gamma(FM) &= A[x := M] \quad \text{if } \mathsf{type}_\Gamma(F) =_\beta \Pi x : \mathsf{type}_\Gamma(M).\, A
\end{aligned}$$

$$\begin{aligned}
\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x) \\
&= \Pi a : *.\, \Pi x : a.\, a \\
&= \Pi a : *.\, a \to a
\end{aligned}$$

$$\begin{aligned}
\mathsf{type}(\Pi a : *.\, a \to a) &= \mathsf{type}_{a:*}(a \to a) \\
&= \mathsf{type}_{a:*}(a) \\
&= *
\end{aligned}$$

computing the type of a term

$$
\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
\mathsf{type}_\Gamma(\Pi x : A.\, B) &= \mathsf{type}_{\Gamma, x:A}(B) \\
\mathsf{type}_\Gamma(A \to B) &= \mathsf{type}_\Gamma(B) \\
\mathsf{type}_\Gamma(*) &= \Box \\
\mathsf{type}_\Gamma(x) &= \Gamma(x) \\
\mathsf{type}_\Gamma(FM) &= A[x := M] \quad \text{if } \mathsf{type}_\Gamma(F) =_\beta \Pi x : \mathsf{type}_\Gamma(M).\, A
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{type}(\lambda a : *.\, \lambda x : a.\, x) &= \Pi a : *.\, \mathsf{type}_{a:*}(\lambda x : a.\, x) \\
&= \Pi a : *.\, \Pi x : a.\, \mathsf{type}_{a:*, x:a}(x) \\
&= \Pi a : *.\, \Pi x : a.\, a \\
&= \Pi a : *.\, a \to a
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{type}(\Pi a : *.\, a \to a) &= \mathsf{type}_{a:*}(a \to a) \\
&= \mathsf{type}_{a:*}(a) \\
&= *
\end{aligned}
$$

the rules of $\lambda 2$

the seven PTS rules:
axiom rule, variable rule, weakening rule, application rule,
abstraction rule, product rule, conversion rule

λP:

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash (\Pi x : A.\, B) : s}$$

$\lambda 2$:

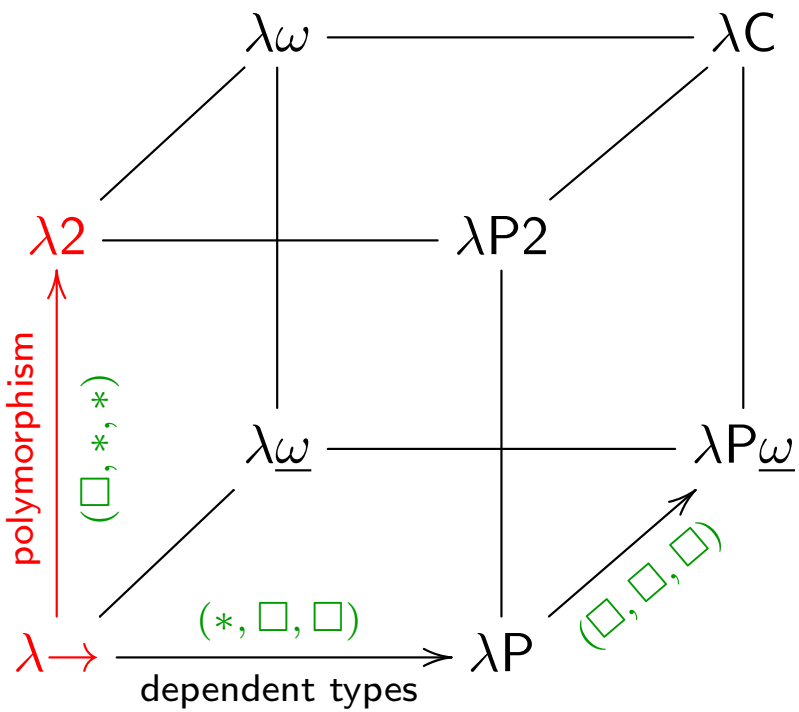
$$\frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : *}{\Gamma \vdash (\Pi x : A.\, B) : *}$$

lambda cube:

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\Pi x : A.\, B) : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

the lambda cube

	$\mathcal{R} =$
$\lambda\!\!\rightarrow$	$\{(*,*,*)\}$
$\lambda 2$	$\{(*,*,*),(\Box,*,*)\}$
λP	$\{(*,*,*),\quad\quad(*,\Box,\Box)\}$
λC	$\{(*,*,*),(\Box,*,*),(*,\Box,\Box),(\Box,\Box,\Box)\}$



examples of types in different systems

$\lambda\!\rightarrow$:

$$(\lambda x : \mathsf{nat}.\, x) : \underbrace{\mathsf{nat}}_{:\,*} \rightarrow \underbrace{\mathsf{nat}}_{:\,*}$$

λP:

$$\mathsf{vec} : \underbrace{\mathsf{nat}}_{:\,*} \rightarrow \underbrace{*}_{:\,\Box}$$

λ2:

$$\mathsf{id} = (\lambda a : *.\, \lambda x : \mathsf{nat}.\, x) : \underbrace{\Pi a : *.\, a}_{:\,\Box} \rightarrow \underbrace{a}_{:\,*}$$

impredicative encoding of datatypes

using the non-dependent recursor as the definition

$$
\begin{aligned}
A \times_2 B &:= \Pi a : *.\, (A \to B \to a) \to a \\
A +_2 B &:= \Pi a : *.\, (A \to a) \to (B \to a) \to a \\
\mathsf{bool}_2 &:= \Pi a : *.\, a \to a \to a \\
\mathsf{nat}_2 &:= \Pi a : *.\, (a \to a) \to a \to a \\
&\ldots
\end{aligned}
$$

the type of polymorphic Church numerals:

$$
\begin{aligned}
3 &:= \lambda a : *.\, \lambda f : a \to a.\, \lambda x : a.\, f(f(fx)) \\
&\vdots \\
\mathsf{nat}_2 &= \Pi a : *.\, (a \to a) \to a \to a
\end{aligned}
$$

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S O)))
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S O)))
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S 0)))
  ============================
  False
```

```
inversion H.
```

```
  H : even (S (S (S 0)))
  n : nat
  H1 : even (S 0)
  H0 : n = S 0
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S O)))
  ============================
  False
```

```
inversion_clear H.
```

```
  H0 : even (S O)
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S 0)))
  ============================
  False
```

```
inversion H.
```

```
  H : even (S (S (S 0)))
  n : nat
  H1 : even (S 0)
  H0 : n = S 0
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S O)))
  ============================
  False
```

```
inversion H; clear H.
```

```
  n : nat
  H1 : even (S O)
  H0 : n = S O
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S O)))
  ============================
  False
```

```
inversion H; clear H; subst.
```

```
  H1 : even (S O)
  ============================
  False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S 0)))
  ============================
  False
```

```
simple inversion H.
```

```
  H : even (S (S (S 0)))
  H0 : 0 = S (S (S 0))
  ============================
  False
```

```
  H : even (S (S (S 0)))
  n : nat
  H1 : S (S n) = S (S (S 0))
  ============================
  even n -> False
```

inversion

when to use inversion

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

```
  H : even (S (S (S O)))
  ============================
  False
```

```
inversion H.
```

```
H : even (S (S (S O)))
n : nat
H1 : even (S O)
H0 : n = S O
============================
False
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

for all parameters of the type,
for all predicates over the type,
if the predicate is preserved by the constructors,
then the predicate holds on the full type

```
even_ind_dep
     :
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

for all parameters of the type,
for all predicates over the type,
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the induction principle of even

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Inductive even : nat -> Prop :=
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```

for all parameters of the type,
for all predicates over the type,
if the predicate is preserved by the constructors,
then the predicate holds on the full type

```
even_ind_dep
     : forall P : forall n : nat, even n -> Prop,
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

for all parameters of the type,
for all predicates over the type,
if the predicate is preserved by the constructors,
then the predicate holds on the full type

```
even_ind_dep
     : forall P : forall n : nat, even n -> Prop,
       P O even_O ->
       (forall (n : nat) (H : even n), P n H ->
         P (S (S n)) (even_SS n H)) ->
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_O : even O
| even_SS n : even n -> even (S (S n)).
```

for all parameters of the type,
for all predicates over the type,
if the predicate is preserved by the constructors,
then the predicate holds on the full type

```
even_ind_dep
     : forall P : forall n : nat, even n -> Prop,
       P O even_O ->
       (forall (n : nat) (H : even n), P n H ->
         P (S (S n)) (even_SS n H)) ->
       forall (n : nat) (H : even n), P n H
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

for all parameters of the type,
for all predicates over the type,
if the predicate is preserved by the constructors,
then the predicate holds on the full type

```
even_ind_dep
     : forall P : forall n : nat, even n -> Prop,
       P 0 even_0 ->
       (forall (n : nat) (H : even n), P n H ->
         P (S (S n)) (even_SS n H)) ->
       forall (n : nat) (H : even n), P n H
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
even_ind_dep
     : forall P : forall n : nat, even n -> Prop,
       P 0 even_0 ->
       (forall (n : nat) (H : even n), P n H ->
         P (S (S n)) (even_SS n H)) ->
       forall (n : nat) (H : even n), P n H
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
even_ind
     : forall P : forall n : nat,           Prop,
       P 0         ->
       (forall (n : nat) (H : even n), P n   ->
         P (S (S n))                ) ->
       forall (n : nat) (H : even n), P n
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
even_ind
     : forall P : forall n : nat, Prop,
       P 0 ->
       (forall (n : nat) (H : even n), P n ->
         P (S (S n)) ->
       forall (n : nat) (H : even n), P n
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

```
even_ind
     : forall P : nat -> Prop,
       P 0 ->
       (forall n : nat, even n -> P n ->
         P (S (S n))) ->
       forall n : nat, even n -> P n
```

the induction principle of even

```
Inductive even : nat -> Prop :=
| even_0 : even 0
| even_SS n : even n -> even (S (S n)).
```

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

```
even_ind
     : forall P : nat -> Prop,
       P 0 ->
       (forall n : nat, even n -> P n ->
         P (S (S n))) ->
       forall n : nat, even n -> P n
```

proving that three is not even

```
1 subgoal (ID 17)

  H : even (S (S (S O)))
  ============================
  False
```

proving that three is not even

```
1 subgoal (ID 17)

  H : even (S (S (S O)))
  ============================
  False
```

$$\mathsf{even}(3) \to \bot$$

proving that three is not even

```
1 subgoal (ID 17)

  H : even (S (S (S O)))
  ============================
  False
```

$$\mathsf{even}(3) \to \bot$$

$$\mathsf{even}(M) \to A$$

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

proving that three is not even

```
1 subgoal (ID 17)

  H : even (S (S (S O)))
  ============================
  False
```

$$\mathsf{even}(3) \to \bot$$

$$\mathsf{even}(M) \to A$$
$$\forall n.\, n = M \to \mathsf{even}(n) \to A$$

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

proving that three is not even

```
1 subgoal (ID 17)

  H : even (S (S (S O)))
  ============================
  False
```

$$\mathsf{even}(3) \to \bot$$

$$\mathsf{even}(M) \to A$$
$$\forall n.\, n = M \to \mathsf{even}(n) \to A$$
$$\forall n.\, \mathsf{even}(n) \to n = M \to A$$

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

proving that three is not even

```
1 subgoal (ID 17)

  H : even (S (S (S O)))
  ============================
  False
```

$$\mathsf{even}(3) \to \bot$$

$$\mathsf{even}(M) \to A$$
$$\forall n.\, n = M \to \mathsf{even}(n) \to A$$
$$\forall n.\, \mathsf{even}(n) \to n = M \to A$$

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

$$P(n) := \big(n = M \to A\big)$$

applying the induction principle

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

$$P(n) := \big(n = 3 \to \bot\big)$$

applying the induction principle

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

$$P(n) := \big(n = 3 \to \bot\big)$$

$$\frac{0 = 3 \to \bot \qquad \forall n.\, \mathsf{even}(n) \to (n = 3 \to \bot) \to n + 2 = 3 \to \bot}{\forall n.\, \mathsf{even}(n) \to n = 3 \to \bot}$$

applying the induction principle

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

$$P(n) := \big(n = 3 \to \bot\big)$$

$$\frac{0 = 3 \to \bot \qquad \forall n.\, \mathsf{even}(n) \to (n = 3 \to \bot) \to n + 2 = 3 \to \bot}{\forall n.\, \mathsf{even}(n) \to n = 3 \to \bot}$$

applying the induction principle

$$\frac{P(0) \qquad \forall n.\, \mathsf{even}(n) \to P(n) \to P(n+2)}{\forall n.\, \mathsf{even}(n) \to P(n)}$$

$$P(n) := \big(n = 3 \to \bot\big)$$

$$\frac{0 = 3 \to \bot \qquad \forall n.\, \mathsf{even}(n) \to (n = 3 \to \bot) \to n + 2 = 3 \to \bot}{\forall n.\, \mathsf{even}(n) \to n = 3 \to \bot}$$

cleaning up the goals

`discriminate`

$$\mathsf{S}\, n \neq \mathsf{O}$$

`injection`

$$\mathsf{S}\, n = \mathsf{S}\, m \to n = m$$

works with all constructors of all inductive types

cleaning up the goals

`discriminate H.`

$$\mathsf{S}\, n \neq \mathsf{O}$$
$$\mathsf{H} : \mathsf{S}\, n = \mathsf{O}$$

`injection H.`

$$\mathsf{S}\, n = \mathsf{S}\, m \to n = m$$
$$\mathsf{H} : \mathsf{S}\, n = \mathsf{S}\, m$$

works with all constructors of all inductive types

how do discriminate and injection work?

```
Definition is_S n :=
  match n with S _ => True | _ => False end.

Definition discriminate_S_O n :
    ~(S n = O) :=
  eq_ind (S n) is_S I O.


Definition S_inv n :=
  match n with S m => m | _ => O end.

Definition injection_S n m :
    S n = S m -> n = m :=
  eq_ind (S n) (fun z => n = S_inv z)
    (eq_refl n) (S m).
```

some tactics

rewriting tactics

```
unfold c.
unfold c in H.
unfold c in *.

simpl.
simpl in H.
simpl in *.


rewrite M.
rewrite <- M.
rewrite M in H.
rewrite M in *.

pattern N at n1 ...nk; rewrite M.

subst.
```

the pattern tactic

```
x, y : A
H : x = y
============================
p x x x
```

the pattern tactic

```
x, y : A
H : x = y
============================
p x x x
```

the pattern tactic

```
  x, y : A
  H : x = y
  ============================
  p x x x
```

```
pattern x at 1 3.
```

```
  x, y : A
  H : x = y
  ============================
  (fun a : A => p a x a) x
```

the pattern tactic

```
  x, y : A
  H : x = y
  ============================
  p x x x
```

```
pattern x at 1 3.
```

```
  x, y : A
  H : x = y
  ============================
  (fun a : A => p a x a) x
```

```
rewrite H.
```

```
  x, y : A
  H : x = y
  ============================
  p y x y
```

elimination tactics

`elim` M`.`
`destruct` M`.`
`induction` x`.`

elimination tactics

elim M.
destruct M.
induction x.

```
H : A \/ B
============================
C
```

elimination tactics

elim M.
destruct M.
induction x.

```
  H : A \/ B
  ============================
  C
```

```
elim H.
```

```
  H : A \/ B                    H : A \/ B
  ============================  ============================
  A -> C                        B -> C
```

elimination tactics

elim M.
destruct M.
induction x.

```
  H : A \/ B
  ============================
  C
```

```
elim H.
```

```
  H : A \/ B                    H : A \/ B
  ============================  ============================
  A -> C                        B -> C
```

$$\frac{\begin{array}{ccc}\vdots & \vdots & \vdots \\ A \vee B & A \to C & B \to C\end{array}}{C}\, E\vee$$

elimination tactics

elim M.
destruct M.
induction x.

```
n : nat
============================
P n
```

elimination tactics

elim M.
destruct M.
induction x.

```
  n : nat
  ============================
  P n
```

```
elim n.
```

```
  n : nat                          n : nat
  ============================     ============================
  P 0                              forall n0 : nat, P n0 -> P (S n0)
```

elimination tactics

elim M.
destruct M.
induction x.

```
  n : nat
  ============================
  P n
```

```
destruct n.
```

```
  ============================   n : nat
  P 0                            ============================
                                 P (S n)
```

elimination tactics

elim M.
destruct M.
induction x.

```
  n : nat
  ============================
  P n
```

```
induction n.
```

```
  ============================
  P O
```

```
  n : nat
  IHn : P n
  ============================
  P (S n)
```

elimination tactics

elim M.
destruct M.
induction x.

```
  n : nat
  ============================
  P n
```

```
elim n.
```

```
  n : nat                          n : nat
  ============================     ============================
  P 0                              forall n0 : nat, P n0 -> P (S n0)
```

elimination tactics

elim M.
destruct M.
induction x.

```
  n : nat
  ============================
  P n
```

```
elim n.
```

```
  n : nat                          n : nat
  ============================     ============================
  P 0                              forall n0 : nat, P n0 -> P (S n0)
```

```
Show Proof.
```

```
(fun n : nat =>
 nat_ind (fun n0 : nat => P n0)
   ?Goal
   ?Goal0 n)
```

elimination tactics

elim M.
destruct M.
induction x.

```
  n : nat
  ============================
  P n
```

```
destruct n.
```

```
  ============================   n : nat
  P O                            ============================
                                 P (S n)
```

```
Show Proof.
```

```
(fun n : nat =>
 match n as n0 return (P n0) with
 | O => ?Goal
 | S n0 => ?Goal0@n:=n0
 end)
```

elimination tactics

```
elim M.
destruct M.
induction x.
```

```
  n : nat
  ============================
  P n
```

```
induction n.
```

```
  ============================
  P 0
```

```
  n : nat
  IHn : P n
  ============================
  P (S n)
```

```
Show Proof.
```

```
(fun n : nat =>
 nat_ind (fun n0 : nat => P n0)
   ?Goal
   (fun (n0 : nat) (IHn : P n0) =>
    ?Goal0@n:=n0) n)
```

induction principle versus fix / match

```
nat_ind =
fun (P : nat -> Prop) (f : P O)
  (f0 : forall n : nat, P n -> P (S n)) =>
fix F (n : nat) : P n :=
  match n as n0 return (P n0) with
  | O => f
  | S n0 => f0 n0 (F n0)
  end
     : forall P : nat -> Prop,
       P O ->
       (forall n : nat, P n -> P (S n)) ->
       forall n : nat, P n
```

induction principle versus fix / match

```
nat_ind =
fun (P : nat -> Prop) (f : P O)
  (f0 : forall n : nat, P n -> P (S n)) =>
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```

program extraction

predecessor with specification

```
Require Import Arith.

Definition pred (n : nat) : lt 0 n -> {m : nat | n = S m}.
intro H. destruct n as [|m].
- elim (lt_irrefl 0 H).
- exists m. reflexivity.
Defined.
```

two inputs: a nat and a proof of `lt 0 n`
two outputs: a nat and a proof of `n = S m`

program extraction

predecessor with specification

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Require Import Arith.

Definition pred (n : nat) : lt 0 n -> {m : nat | n = S m}.
intro H. destruct n as [|m].
- elim (lt_irrefl 0 H).
- exists m. reflexivity.
Defined.
```

two inputs: a nat and a proof of `lt 0 n`
two outputs: a nat and a proof of `n = S m`

the output type is a Sigma type of dependent pairs:

```
{m : nat | n = S m}
        ||
@sig nat (fun m : nat => n = S m)
```

the Coq term for predecessor with specification

```
pred =
fun (n : nat) (H : lt 0 n) =>
match n as n0 return (lt 0 n0 -> {m : nat | n0 = S m}) with
| 0 =>
    fun H0 : lt 0 0 =>
    False_rec {m : nat | 0 = S m} (lt_irrefl 0 H0)
| S m =>
    fun _ : lt 0 (S m) =>
    exist (fun m0 : nat => S m = S m0) m eq_refl
end H
     : forall n : nat, lt 0 n -> {m : nat | n = S m}
```

extracting predecessor

```
Extraction pred.
```

```
(** val pred : nat -> nat **)

let pred = function
| O -> assert false (* absurd case *)
| S m -> m
```

extracting predecessor

```
Recursive Extraction pred.
```

```
type 'a sig0 = 'a
  (* singleton inductive, whose constructor was exist *)

type nat =
| O
| S of nat

(** val pred : nat -> nat **)

let pred = function
| O -> assert false (* absurd case *)
| S m -> m
```

extracting predecessor

```
Recursive Extraction pred.
```

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type 'a sig0 = 'a
  (* singleton inductive, whose constructor was exist *)

type nat =
| O
| S of nat

(** val pred : nat -> nat **)

let pred = function
| O -> assert false (* absurd case *)
| S m -> m
```

removes all objects of which the type is in Prop
removes all the dependencies from the types

logical frameworks

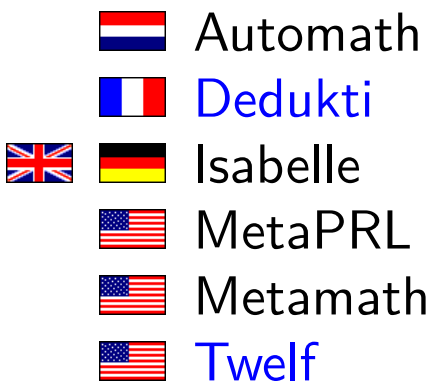
systems for multiple logics

- **most proof assistants**

  logic: fixed
  theory: defined

- **logical frameworks**

  logic: defined
  theory: defined



two kinds of Curry-Howard

- **propositions are types**

  single logic for each type theory

$$A : *$$
$$M : A$$

- **propositions are objects**

  different logics possible

$$\mathsf{form} : *$$
$$\mathsf{True} : \mathsf{form} \to *$$

$$A : \mathsf{form}$$
$$M : \mathsf{True}\ A$$

a λP context for a small logic

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}\, I\wedge \qquad \frac{\vdots}{A \wedge B}\, El\wedge \qquad \frac{\vdots}{A \wedge B}\, Er\wedge$$

a λP context for a small logic

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}\, I\wedge \qquad \frac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{A}\, El\wedge \qquad \frac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{B}\, Er\wedge$$

$$\begin{array}{l}
\mathsf{form} : * \\
\quad \wedge : \mathsf{form} \to \mathsf{form} \to \mathsf{form} \\
\\
\mathsf{True} : \mathsf{form} \to * \\
\quad I\wedge : \Pi A : \mathsf{form}.\, \Pi B : \mathsf{form}.\, \mathsf{True}\ A \to \mathsf{True}\ B \to \mathsf{True}\ (\wedge\ A\ B) \\
El\wedge : \Pi A : \mathsf{form}.\, \Pi B : \mathsf{form}.\, \mathsf{True}\ (\wedge\ A\ B) \to \mathsf{True}\ A \\
El\wedge : \Pi A : \mathsf{form}.\, \Pi B : \mathsf{form}.\, \mathsf{True}\ (\wedge\ A\ B) \to \mathsf{True}\ B
\end{array}$$

impredicativity

a $\lambda 2$ typing judgment

$$b : * \vdash (\Pi a : *.\ a \to b) : \ldots ?$$

impredicativity

a $\lambda 2$ typing judgment

$$b : * \vdash (\Pi a : *.\ a \to b) : \ldots ?$$

$$\begin{aligned}
\mathsf{type}_\Gamma(\lambda x : A.\, M) &= \Pi x : A.\, \mathsf{type}_{\Gamma, x:A}(M) \\
\mathsf{type}_\Gamma(\Pi x : A.\, B) &= \mathsf{type}_{\Gamma, x:A}(B) \\
\mathsf{type}_\Gamma(A \to B) &= \mathsf{type}_\Gamma(B)
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impredicativity

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the problem with a set theoretic interpretation

$$b : * \vdash (\Pi a : *.\, a \to b) : *$$

the problem with a set theoretic interpretation

$$\mathsf{bool} : * \vdash (\Pi a : *.\, a \to \mathsf{bool}) : *$$

the problem with a set theoretic interpretation

$$\mathsf{bool} : * \vdash (\Pi a : *.\, \underbrace{a \to \mathsf{bool}}) : *$$

the power set of a

the problem with a set theoretic interpretation

$$\mathsf{bool} : * \vdash (\Pi a : *.\, \underbrace{a \to \mathsf{bool}}_{\text{the power set of } a}) : *$$

the power set of a

$$\prod_{a \in \mathsf{Set}} \overbrace{\mathcal{P}(a)} \in \mathsf{Set}$$

the problem with a set theoretic interpretation

$$\mathsf{bool} : * \vdash (\Pi a : *.\, \underbrace{a \to \mathsf{bool}}_{\text{the power set of } a}) : *$$

$$X := \prod_{a \in \mathsf{Set}} \overbrace{\mathcal{P}(a)}^{\text{the power set of } a} \in \mathsf{Set}$$

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$$X = \mathcal{P}(X) \times \prod_{\substack{a \in \mathsf{Set} \\ a \neq X}} \mathcal{P}(a)$$

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but: X has a smaller cardinality than $\mathcal{P}(X)$!

diagonalisation

$$X := (\Pi a : *.\, a \to \mathsf{bool}) : *$$

diagonalisation

$$X := (\Pi a : *.\, a \to \mathsf{bool}) : *$$

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$$f : X$$
$$f : \Pi a : *.\, a \to \mathsf{bool}$$

diagonalisation

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$$\begin{aligned} f &: X \\ f &: \Pi a : *.\, a \to \mathsf{bool} \\ fX &: X \to \mathsf{bool} \end{aligned}$$

diagonalisation

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$$fXf = \neg(fXf)$$

diagonalisation

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$$\begin{aligned} fXf &= \neg(fXf) \\ \mathsf{true} &= \mathsf{false} \end{aligned}$$

needs if-then-else on equality of types
and proof irrelevance for equality

impredicativity in Coq

Set is predicative
Prop is impredicative

```
Inductive bool : Set := true : bool | false : bool.
Check (forall a : Set, a -> bool).
```

```
forall a : Set, a -> bool
     : Type
```

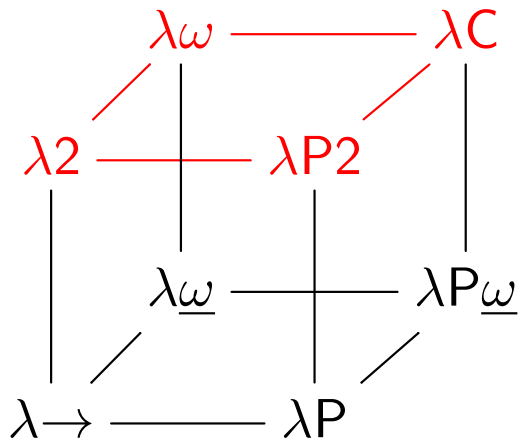
$$b : * \vdash (\Pi a : *.\ a \to b) : \Box$$

```
Inductive pbool : Prop := ptrue : pbool | pfalse : pbool.
Check (forall a : Prop, a -> pbool).
```

```
forall a : Prop, a -> pbool
     : Prop
```

$$b : * \vdash (\Pi a : *.\ a \to b) : *$$

the impredicative plane of the lambda cube



impredicativity = defining something by quantifying over a domain that contains the thing you are defining

generally not inconsistent with classical mathematics

you often define the smallest set closed under some operations as the intersection of all such sets

conclusion

summary of Femke's part of the course

- untyped lambda calculus

$$M, N ::= x \mid MN \mid \lambda x.\, M$$

$$(\lambda x.\, M)N \to_\beta M[x := N]$$

- typed lambda calculus = type theories

$$\Gamma \vdash M : A$$

- Curry-Howard correspondence
  proof terms

propositional logic	STT	simple types
predicate logic	λP	dependent types
second order logic	$\lambda 2$	polymorphic types

summary of the first part of the course (continued)

- CIC = the type theory of Coq
  inductive types

  induction principles = recursion principles

  $$M \to_{\beta\delta\iota\zeta\eta} N$$

- Coq as a functional programming language

  `Inductive`
  `Fixpoint match`

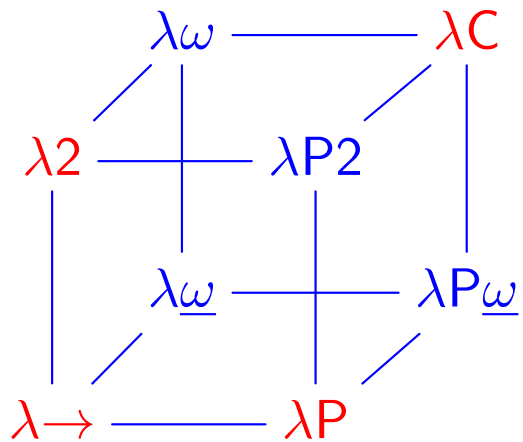
- Coq as a proof language

  `Lemma Definition Qed Defined`
  `intros apply`
  `reflexivity split left right exists`
  `elim destruct induction inversion`
  `unfold simpl compute rewrite pattern clear subst`
  `Check Print Show Eval`

thank you

thanks for listening!

questions?



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