Type Theory and Coq

Exercises on Normalization

1. In the proof of WN for $\lambda\!\to$, the height of a type $h(\sigma)$ is defined by

   - $h(\alpha) := 0$
   - $h(\sigma_1 \to \ldots \to \sigma_n \to \alpha) := \max(h(\sigma_1), \ldots, h(\sigma_n)) + 1.$

   Prove that this is the same as taking as the second clause

   - $h(\sigma \to \tau) := \max(h(\sigma) + 1, h(\tau)).$

2. Consider the following term $N : A$, where $A = \alpha \to \alpha$ and $\mathbf{I}_1 : A$ and $\mathbf{I}_2 : A \to A$ and $\mathbf{I}_3 : A$ and $\mathbf{I}_4 : A$ are copies of the well-known λ-term $\mathbf{I}$ ($:= \lambda x.x$).

$$N \quad := \quad \lambda y{:}\alpha.(\lambda x{:}A \to A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

   (a) Determine $m(N)$, the *measure* of N as defined in the weak normalization proof.

   (b) Determine which redex will be contracted following the strategy in the weak normalization proof, obtaining a term N'.

   (c) Determine $m(N')$, the measure of this reduct of N.

3. In the proof of WN for $\lambda\!\to$, it is stated that, if $M \longrightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that does not contain any other redex of height $h(M)$, then this does not create a new redex of maximum height.

   Show that this holds for the case

$$\begin{array}{rcl} M & = & (\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I}))(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I})) \\ & \longrightarrow_\beta & (\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))(\lambda v : B.(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))\,\mathbf{I}) \quad = \quad P \end{array}$$

   where $B = \alpha \to \alpha$, $C = B \to B$ and $A = C \to B$.

   Also show that $m(M) >_l m(P)$.

4. Suppose X, Y, and Z are properties of λ-terms. Then we can have the following situations: If M satisfies property X and N satisfies property Y, then

   (a) Yes, property Z always holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \in Z)$

   (b) No, property Z never holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \notin Z)$

   (c) Undec, property Z holds for some M, N, and doesn't hold for some other M, N (so $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \in Z$ and $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \notin Z)$

Fill in the following diagram with Yes, No and Undec and motivate your answers. In case of Undec, give M, N for both cases.

	$N \in \mathsf{WN}$	$N \in \neg\mathsf{SN}$
$M \in \mathsf{WN}$	$M\,N \in \mathsf{WN}$?	$M\,N \in \mathsf{SN}$?
$M \in \mathsf{SN}$	$M\,N \in \mathsf{SN}$?	$M\,N \in \neg\mathsf{SN}$?

5. Prove that *type reduction* is SN for $\lambda 2$ a la Church. (Define a simple measure on terms that decreases with type reduction.)