Type Theory and Coq
Exercises on Normalization

1. In the proof of WN for $\lambda\!\to$, the height of a type $h(\sigma)$ is defined by

   - $h(\alpha) := 0$
   - $h(\sigma_1 \to \ldots \to \sigma_n \to \alpha) := \max(h(\sigma_1), \ldots, h(\sigma_n)) + 1.$

   Prove that this is the same as taking as the second clause

   - $h(\sigma \to \tau) := \max(h(\sigma) + 1, h(\tau)).$

   Answer: ....
   Recall that $\sigma_1 \to \ldots \sigma_n \to \alpha$ should be read as $(\sigma_1 \to (\ldots (\sigma_n \to \alpha) \ldots)$ and this measn that every type $\sigma \to \tau$ can be written as $\sigma \to \tau_1 \ldots \tau_n \to \alpha$ for some α.

   We have two definitions of h that we call h_1 and h_2 for now:

   - $h_1(\alpha) := 0,$
   - $h_1(\sigma_1 \to \ldots \to \sigma_n \to \alpha) := \max(h_1(\sigma_1), \ldots, h_1(\sigma_n)) + 1.$
   - $h_2(\alpha) := 0,$
   - $h_2(\sigma \to \tau) := \max(h_2(\sigma) + 1, h_2(\tau)).$

   We prove that $h_1(\sigma) = h_2(\sigma)$ for all σ by induction on σ. For type variables this is the case. Case $\sigma = \sigma_1 \to \ldots \to \sigma_n \to \alpha$:

   $$\begin{aligned} h_2(\sigma) &= \max(h_2(\sigma_1) + 1, h_2(\sigma_2 \to \ldots \to \sigma_n \to \alpha)) \\ &\overset{\text{IH}}{=} \max(h_1(\sigma_1) + 1, h_1(\sigma_2 \to \ldots \to \sigma_n \to \alpha)) \\ &= \max(h_1(\sigma_1) + 1, \max(h_1(\sigma_2), \ldots, h_1(\sigma_n)) + 1) \\ &= \max(h_1(\sigma_1), \ldots, h_1(\sigma_n)) + 1 \\ &= h_1(\sigma) \end{aligned}$$

   End Answer ....

2. Consider the following term $N : A$, where $A = \alpha \to \alpha$ and $\mathbf{I}_1 : A$ and $\mathbf{I}_2 : A \to A$ and $\mathbf{I}_3 : A$ and $\mathbf{I}_4 : A$ are copies of the well-known λ-term $\mathbf{I}$ $(:= \lambda x.x)$.

   $$N \quad := \quad \lambda y{:}\alpha.(\lambda x{:}A \to A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$$

   (a) Determine $m(N)$, the *measure* of N as defined in the weak normalization proof.
   Answer: ....
   There are three redexes:

- $R_1 := \mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y))$ of height the type of $\mathbf{I}_1$, which is $h(A) = 1$.
- $R_2 := (\lambda x{:}A{\to}A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y)))\,\mathbf{I}_2$ of height the type of $\lambda x{:}A{\to}A.\mathbf{I}_1\,(x\,\mathbf{I}_4\,(\mathbf{I}_3\,y))$, which is $h((A{\to}A){\to}\alpha) = 3$.
- $R_3 := \mathbf{I}_3\,y$ of height the type of $\mathbf{I}_3$, which is $h(A) = 1$.

NB. the sub-term $\mathbf{I}_4\,(\mathbf{I}_3\,y)$ is not a redex, as a matter of fact, it isn't a sub-term as the brackets should be read as follows: $(x\,\mathbf{I}_4)\,(\mathbf{I}_3\,y)$!
So $m(N) = (3,1)$.
End Answer . . . .

(b) Determine which redex will be contracted following the strategy in the weak normalization proof, obtaining a term N'.
Answer: . . . .
We contract redex R_2, obtaining

$$N' := \lambda y{:}\alpha.\mathbf{I}_1\,(\mathbf{I}_2\,\mathbf{I}_4\,(\mathbf{I}_3\,y))$$

End Answer . . . .

(c) Determine $m(N')$, the measure of this reduct of N.
Answer: . . . .
N' still has redexes R_1 and R_3 that have height 1. We have a new redex $\mathbf{I}_2\,\mathbf{I}_4$ whose height is $h(A{\to}A)$ (the height of the type of $\mathbf{I}_2$), which is 2. So $m(N') = (2,1)$
End Answer . . . .

3. In the proof of WN for $\lambda{\to}$, it is stated that, if $M \longrightarrow_\beta N$ by contracting a redex of maximum height, $h(M)$, that does not contain any other redex of height $h(M)$, then this does not create a new redex of maximum height.

   Show that this holds for the case

$$\begin{array}{rcll} M & = & (\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I}))(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I})) & \\ & \longrightarrow_\beta & (\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))(\lambda v : B.(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))\,\mathbf{I}) & = \quad P \end{array}$$

   where $B = \alpha{\to}\alpha$, $C = B{\to}B$ and $A = C{\to}B$.

   Also show that $m(M) >_l m(P)$.
   Answer: . . . .
   In M there are two redexes:

   - $R_1 := \mathbf{I}\,\mathbf{I}$, whose height is the height of the type of the $\mathbf{I}$ on the left, which is C. So the height of R_1 is $h(C)$.
   - $R_2 := (\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I}))(\lambda z : C.z\,(\mathbf{I}\,\mathbf{I}))$, the whole term, whose height is the height of the type of $\lambda x : A.x\,(\lambda v : B.x\,\mathbf{I})$, which is $A{\to}B$. So the height of R_2 is $h(A{\to}B)$.

   $h(B) = 1$, $h(C) = 2$ and $h(A) = 3$. Redex R_2 is the single redex of maximum height, which is 4, so $m(M) = (4,1)$. Note that redex R_1 has height $h(C)$ which is 2.

In N we have two copies of redex R_1 (of height 2) and two new redexes whose height is the height of the type of $\lambda z : C.z\,(\mathbf{I}\,\mathbf{I})$, which is $C{\to}B$. We have $h(C{\to}B) = 3$, so we have $m(N) = (3, 2)$. We see that $m(M) = (4, 1) >_l (3, 2) = m(P)$.
End Answer ....

4. Suppose X, Y, and Z are properties of λ-terms. Then we can have the following situations: If M satisfies property X and N satisfies property Y, then

   (a) Yes, property Z always holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \in Z)$

   (b) No, property Z never holds (so $\forall M, N(M \in X \wedge N \in Y \Rightarrow M\,N \notin Z)$

   (c) Undec, property Z holds for some M, N, and doesn't hold for some other M, N (so $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \in Z$ and $\exists M, N(M \in X \wedge N \in Y \wedge M\,N \notin Z)$

   Fill in the following diagram with Yes, No and Undec and motivate your answers. In case of Undec, give M, N for both cases.

	$N \in \mathsf{WN}$	$N \in \neg\mathsf{SN}$
$M \in \mathsf{WN}$	$M\,N \in \mathsf{WN}$?	$M\,N \in \mathsf{SN}$?
$M \in \mathsf{SN}$	$M\,N \in \mathsf{SN}$?	$M\,N \in \neg\mathsf{SN}$?

Answer: ....
We have

	$N \in \mathsf{WN}$		$N \in \neg\mathsf{SN}$	
$M \in \mathsf{WN}$	$M\,N \in \mathsf{WN}$	Undec[1]	$M\,N \in \mathsf{SN}$	No[2]
$M \in \mathsf{SN}$	$M\,N \in \mathsf{SN}$	Undec[3]	$M\,N \in \neg\mathsf{SN}$	Yes[4]

For Undec[1] and Undec[3] consider $M, N = \lambda x.\,x\,x$ for a negative example and $M, N = \lambda x.\,x$ for a positive example.

For No[2] and Yes[4], if $N \in \neg\mathsf{SN}$, then N has an infinite reduction path, so $M\,N$ has an infinite reduction path (no matter what M is).
End Answer ....

5. Prove that *type reduction* is SN for $\lambda 2$ a la Church. (Define a simple measure on terms that decreases with type reduction.)
Answer: ....
Define the measure $f(M)$ as $f(M) :=$ the number of type-abstractions in M, so recursively:

$$\begin{array}{rcl} f(x) & := & 0 \\ f(P\,N) & := & f(P) + f(N) \\ f(\lambda\alpha.\,N) & := & 1 + f(N) \end{array}$$

$$\begin{array}{rcl} f(P\,\sigma) & := & f(P) \\ f(\lambda x:\sigma.\,N) & := & f(N) \\ f(P\,N) & := & f(P) + f(N) \end{array}$$

Then we have that if $M \longrightarrow_\beta N$ by a type-reduction (i.e. a redex contraction of the form $(\lambda\alpha.P)\sigma \longrightarrow_\beta P[\alpha := \sigma]$), then $f(M) > f(N)$, which is easily proved by induction on M.

So: type-reduction is strongly normalizing.
End Answer . . . .