Finitary Higher Inductive Types in the Groupoid Model - 2-Hits

Based on the paper by Peter Dybjer and Hugo Moeneclaey, 2018

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Basic goal

In a 1-HIT, we need introduction rules for two kinds of objects:

- **Points**: Basic elements of the type.
- **Paths**: Higher-dimensional constructs representing equality between *points*.

A 2-Hit needs these as well as introduction rules for a third kind of object:

- **Surfaces**: Higher-dimensional constructs representing equality between *paths*.

Reminder: Notation

- $I(A, a, a')$ will denote the *general identity former*, elements of which are proofs (or paths) that a and a' are equal elements of type A
- Instead of $I(A, a, a')$, we will write

$$a =_A a'.$$

- For a *type family* $B : A \to \mathrm{Type}$, we will denote the *heterogeneous identity type* by

$$b =_g^B b' := (\mathrm{tr}_g^B(b) = b'),$$

  where $b : B(a)$, $b' : B(a')$ and $g : a =_A a'$
- For a function $f : (x : A) \to B(x)$, we have

$$\mathbf{apd}_f(g : x =_A x') : f(x) =_g^B f(x').$$

(Simplified) form of constructors

Let H be a 2-Hit.

- **Point constructors** (same as 1-HIT)

$$c_0 : A \longrightarrow H \longrightarrow H$$

- **Path constructors** (same as 1-HIT)

$$c_1 : (x : B) \longrightarrow (y : H) \longrightarrow p_1(x, y) =_H q_1(x, y) \longrightarrow p_2(x, y) =_H q_2(x, y)$$

- **Surface constructors**

$$\begin{aligned} c_2 : (x : D) \longrightarrow (y : H) \longrightarrow (z : p_3(x, y) =_H q_3(x, y)) \\ \longrightarrow g_1(x, y, z) =_{p_4(x,y) =_H q_4(x,y)} h_1(x, y, z) \\ \longrightarrow g_2(x, y, z) =_{p_5(x,y) =_H q_5(x,y)} h_2(x, y, z) \end{aligned}$$

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Here, p_i, q_i are *point constructor patterns* with syntax

$$p ::= y \mid c_0(a, p)$$

and g_i, h_i are *path constructor patterns* with syntax

$$g ::= z \mid c_1(b, p, g) \mid g \circ g \mid \mathrm{id} \mid g^{-1}$$

Operations on paths

- $$\frac{g : x = y \qquad h : y = z}{g \circ h : x = z}$$

- $$\frac{}{\mathrm{id} : x = x}$$

- $$\frac{g : x = y}{g^{-1} : y = x}$$

Example: the 2-sphere as a 2-Hit

We define the 2-sphere $\mathbb{S}^2$ inductively:

- **Point constructor**

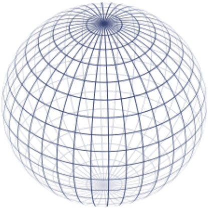
$$\text{base} : \mathbb{S}^2$$

- **Path constructors**

None!

- **Surface constructor**

$$\text{surf} : \text{refl}_{\text{base}} =_{\text{base} =_{\mathbb{S}^2} \text{base}} \text{refl}_{\text{base}}$$



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In this case:

- $c_0 : A \longrightarrow H \longrightarrow H$
- $c_2 : \ (x : D) \longrightarrow (y : H) \longrightarrow (z : p_3(x,y) =_H q_3(x,y))$

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Example: the torus as a 2-Hit

We define the torus T^2 inductively:

- **Point constructor**

$$\mathrm{base} : T^2$$

- **Path constructors**

$$\mathrm{path}_1 : \mathrm{base} =_{T^2} \mathrm{base}$$

$$\mathrm{path}_2 : \mathrm{base} =_{T^2} \mathrm{base}$$

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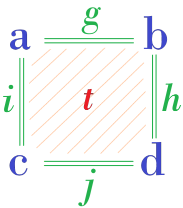

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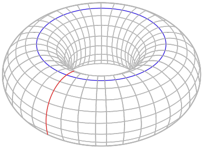

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Example: free groups as a 2-Hit

Let A be a set. The free group $F(A)$ can be defined as a higher inductive type with the following constructors:

- **Point constructors**

$$\eta : A \to F(A)$$

$$m : F(A) \to F(A) \to F(A)$$

$$e : F(A)$$

$$i : F(A) \to F(A)$$

- **Path constructors**

$$\text{assoc} : \forall x, y, z : F(A) \,.\, m(x, m(y, z)) = m(m(x, y), z)$$

$$\text{unit}_1 : \forall x : F(A) \,.\, m(x, e) = x$$

$$\text{unit}_2 : \forall x : F(A) \,.\, m(e, x) = x$$

$$\text{inv}_1 : \forall x : F(A) \,.\, m(x, i(x)) = e$$

$$\text{inv}_2 : \forall x : F(A) \,.\, m(i(x), x) = e$$

- **Surface constructor**

$$\text{trunc} : \forall x, y : F(A) \,.\, \forall p, q : x =_{F(A)} y \,.\, p =_{x =_{F(A)} y} q$$

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Elimination Rules

Goal: Define a function $f : (x : H) \to C(x)$

Assume we have *step functions* $\widetilde{c_0}$ and $\widetilde{c_1}$ (from last presentation):

- $$\widetilde{c_0} : (x : A) \longrightarrow (y : H) \longrightarrow C(y) \longrightarrow C(c_0(x,y))$$
- $$\begin{aligned} \widetilde{c_1} : (x : B) \longrightarrow (y : H) \longrightarrow (\widetilde{y} : C(y)) \\ \longrightarrow (z : p_1 =_H q_1) \longrightarrow T_0(p_1) =^C_z T_0(q_1) \\ \longrightarrow T_0(p_2) =^C_{c_1(x,y,z)} T_0(q_2) \end{aligned}$$

and moreover

- $$\begin{aligned} \widetilde{c_2} : (x : D) \longrightarrow (y : H) \longrightarrow (\widetilde{y} : C(y)) \\ \longrightarrow (z : p_3 =_H q_3) \longrightarrow (\widetilde{z} : T_0(p_3) =^C_z T_0(q_3)) \\ \longrightarrow (t : g_1 =_{p_4 =_H q_4} h_1) \longrightarrow T_1(g_1) =_t^{T_0(p_4) =^H_{-} T_0(q_4)} T_1(h_1) \\ \longrightarrow T_1(g_2) =^{T_0(p_5) =^H_{-} T_0(q_5)}_{c_2(x,y,z,t)} T_1(h_2) \end{aligned}$$

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$$\begin{aligned} T_0(p) : C(p) &= \textit{lifting of } p : H \\ T_1(g) : T_0(p) =^C_g T_0(q) &= \textit{lifting of } g : p =_H q \end{aligned}$$

Lifting

Lifting of a *point constructor pattern* $p : H$ by induction on the form of point constructor patterns:

$$p ::= y \mid c_0(a, p)$$

- $T_0(y) = \tilde{y}$
- $T_0(c_0(a, p)) = \tilde{c_0}(a, p, T_0(p))$

How to lift a *path constructor pattern* $g : p =_H q$ by induction on the form of path constructor patterns:

$$g ::= z \mid c_1(b, p, g) \mid g \circ g \mid \mathrm{id} \mid g^{-1}$$

- $T_1(z) = \tilde{z}$
- $T_1(c_1(b, p, g)) = \tilde{c_1}(b, p, T_0(p), g, T_1(g))$
- $T_1(g \circ g') = T_1(g) \circ' T_1(g')$
- $T_1(\mathrm{id}) = \mathrm{id}$
- $T_1(g^{-1}) = T_1(g)^{-1}$

Operations on liftings of paths

- $$\frac{g : x = y \qquad h : y = z}{g \circ h : x = z}$$
  $$\frac{\bar{g} : \bar{x} =_g \bar{y} \qquad \bar{h} : \bar{y} =_h \bar{z}}{\bar{g} \circ' \bar{h} : \bar{x} =_{g \circ h} \bar{z}}$$

- $$\frac{}{\mathrm{id} : x = x}$$
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- $$\frac{g : x = y}{g^{-1} : y = x}$$
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Equality Rules

Goal: Define a function $f : (x : H) \to C(x)$

The equality rules for f are:

- $f(c_0(x, y)) = \widetilde{c_0}(x, y, f(y))$
- $\mathbf{apd}_f(c_1(x, y, z)) = \widetilde{c_1}(x, y, f(y), z, \mathbf{apd}_f(z))$
- $\mathbf{apd}^2_f(c_2(x, y, z, t)) = \widetilde{c_2}(x, y, f(y), z, \mathbf{apd}_f(z), t, \mathbf{apd}^2_f(t))$

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But: what is $\mathbf{apd}^2_f$?

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Reminder: Let B be a type family over A. Let $x, x' : A$. For a function $f : (x : A) \to B(x)$, we have

$$\boldsymbol{apd}_f(g : x =_A x') : f(x) =^B_g f(x').$$

Let B be a type family over A. Let $x, x' : A$ and $g, h : x =_H x'$. For a function $f : (x : A) \to B(x)$, we have

$$\mathbf{apd}^2_f(r : g =_{x =_H x'} h) : \mathbf{apd}_f(g) =^B_r \mathbf{apd}_f(h).$$

Elimination Rules

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Back to the example of the 2-sphere

Two constructors of the 2-sphere $\mathbb{S}^2$:

- base : $\mathbb{S}^2$
- surf : $\text{refl}_{\text{base}} =_{\text{base} =_{\mathbb{S}^2} \text{base}} \text{refl}_{\text{base}}$

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Thus:

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- surf : $\mathrm{refl}_{\mathrm{base}} =_{\mathrm{base} =_{\mathbb{S}^2} \mathrm{base}} \mathrm{refl}_{\mathrm{base}}$

Thus:

- $\widetilde{c_0} : C(c_0(x, y))$
- $\widetilde{c_2} : T_1(g_2) =^{T_0(p_5) =^{H}_{-} T_0(q_5)}_{c_2(x,y,z,t)} T_1(h_2)$

Elimination of the 2-sphere

Two constructors of the 2-sphere $\mathbb{S}^2$:

- base : $\mathbb{S}^2$
- surf : $\mathrm{refl}_{\mathrm{base}} =_{\mathrm{base} =_{\mathbb{S}^2} \mathrm{base}} \mathrm{refl}_{\mathrm{base}}$

Thus:

- $\widetilde{\mathrm{base}} : C(base)$
- $\widetilde{\mathrm{surf}} : T_1(\mathrm{refl}_{\mathrm{base}}) =^{T_0(\mathrm{base}) =^{C}_{-} T_0(\mathrm{base})}_{\mathrm{surf}} T_1(\mathrm{refl}_{\mathrm{base}})$

Elimination of the 2-sphere

Two constructors of the 2-sphere $\mathbb{S}^2$:

- $\mathrm{base} : \mathbb{S}^2$
- $\mathrm{surf} : \mathrm{refl}_{\mathrm{base}} =_{\mathrm{base} =_{\mathbb{S}^2} \mathrm{base}} \mathrm{refl}_{\mathrm{base}}$

Thus:

- $\widetilde{\mathrm{base}} : C(base)$
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Equality rules for the 2-sphere

Assume we have

- $\widetilde{\text{base}} : C(\text{base})$
- $\widetilde{\text{surf}} : \widetilde{\text{refl}_{\widetilde{\text{base}}}} =^{\widetilde{\text{base}} =^{C}_{-} \widetilde{\text{base}}}_{\text{surf}} \widetilde{\text{refl}_{\widetilde{\text{base}}}}$

Then there exists a function $f : (x : \mathbb{S}^2) \to C(x)$ such that

- $f(\text{base}) = \widetilde{\text{base}}$
- $\mathbf{apd}^2_f(\text{surf}) = \widetilde{\text{surf}}$

Back to the example of the torus

Four constructors of the torus T^2:

- base : T^2
- path_1 : $\text{base} =_{T^2} \text{base}$
- path_2 : $\text{base} =_{T^2} \text{base}$
- surf : $\text{path}_1 \circ \text{path}_2 =_{\text{base}=_{T^2}\text{base}} \text{path}_2 \circ \text{path}_1$

Back to the example of the torus

Four constructors of the torus T^2:

- base : T^2
- $\text{path}_1 : \text{base} =_{T^2} \text{base}$
- $\text{path}_2 : \text{base} =_{T^2} \text{base}$
- surf : $\text{path}_1 \circ \text{path}_2 =_{\text{base} =_{T^2} \text{base}} \text{path}_2 \circ \text{path}_1$

In this case:

- $c_0 : A \longrightarrow H \longrightarrow H$
- $c_1 : (x : B) \longrightarrow (y : H) \longrightarrow p_1(x, y) =_H q_1(x, y) \longrightarrow p_2(x, y) =_H q_2(x, y)$
- $c_2 : (x : D) \longrightarrow (y : H) \longrightarrow (z : p_3(x, y) =_H q_3(x, y))$

  $\longrightarrow g_1(x, y, z) =_{p_4(x,y) =_H q_4(x,y)} h_1(x, y, z)$

  $\longrightarrow g_2(x, y, z) =_{p_5(x,y) =_H q_5(x,y)} h_2(x, y, z)$

Elimination of the torus

Four constructors of the torus T^2:

- base : T^2
- path_1 : base $=_{T^2}$ base
- path_2 : base $=_{T^2}$ base
- surf : $\text{path}_1 \circ \text{path}_2 =_{\text{base}=_{T^2}\text{base}} \text{path}_2 \circ \text{path}_1$

Thus:

- $\widetilde{c_0} : (x : A) \longrightarrow (y : H) \longrightarrow C(y) \longrightarrow C(c_0(x, y))$
- $$\widetilde{c_1} : (x : B) \longrightarrow (y : H) \longrightarrow (\widetilde{y} : C(y)) \longrightarrow (z : p_1 =_H q_1) \longrightarrow T_0(p_1) =^C_z T_0(q_1) \longrightarrow T_0(p_2) =^C_{c_1(x,y,z)} T_0(q_2)$$
- $$\widetilde{c_2} : (x : D) \longrightarrow (y : H) \longrightarrow (\widetilde{y} : C(y)) \longrightarrow (z : p_3 =_H q_3) \longrightarrow (\widetilde{z} : T_0(p_3) =^C_z T_0(q_3)) \longrightarrow (t : g_1 =_{p_4 =_H q_4} h_1) \longrightarrow T_1(g_1) =^{T_0(p_4) =^H_- T_0(q_4)}_t T_1(h_1) \longrightarrow T_1(g_2) =^{T_0(p_5) =^H_- T_0(q_5)}_{c_2(x,y,z,t)} T_1(h_2)$$

Elimination of the torus

Four constructors of the torus T^2:

- base : T^2
- path_1 : base $=_{T^2}$ base
- path_2 : base $=_{T^2}$ base
- surf : $\text{path}_1 \circ \text{path}_2 =_{\text{base} =_{T^2} \text{base}} \text{path}_2 \circ \text{path}_1$

Thus:

- $\widetilde{c_0} : C(c_0(x, y))$
- $\widetilde{c_1} : T_0(p_2) =^{C}_{c_1(x,y,z)} T_0(q_2)$
- $\widetilde{c_2} : T_1(g_2) =^{T_0(p_5) =^{H}_{-} T_0(q_5)}_{c_2(x,y,z,t)} T_1(h_2)$

Elimination of the torus

Four constructors of the torus T^2:

- base : T^2
- path_1 : $\text{base} =_{T^2} \text{base}$
- path_2 : $\text{base} =_{T^2} \text{base}$
- surf : $\text{path}_1 \circ \text{path}_2 =_{\text{base}=_{T^2}\text{base}} \text{path}_2 \circ \text{path}_1$

Thus:

- $\widetilde{\text{base}} : C(\text{base})$
- $\widetilde{\text{path}_1} : T_0(\text{base}) =^{C}_{\text{path}_1} T_0(\text{base})$
- $\widetilde{\text{path}_2} : T_0(\text{base}) =^{C}_{\text{path}_2} T_0(\text{base})$
- $\widetilde{\text{surf}} : T_1(\text{path}_1 \circ \text{path}_2) =^{T_0(\text{base}) =^{C}_{-} T_0(\text{base})}_{\text{surf}} T_1(\text{path}_2 \circ \text{path}_1)$

Elimination of the torus

Four constructors of the torus T^2:

- base : T^2
- path_1 : $\text{base} =_{T^2} \text{base}$
- path_2 : $\text{base} =_{T^2} \text{base}$
- surf : $\text{path}_1 \circ \text{path}_2 =_{\text{base} =_{T^2} \text{base}} \text{path}_2 \circ \text{path}_1$

Thus:

- $\widetilde{\text{base}}$: $C(\text{base})$
- $\widetilde{\text{path}_1}$: $T_0(\text{base}) =^C_{\text{path}_1} T_0(\text{base})$
- $\widetilde{\text{path}_2}$: $T_0(\text{base}) =^C_{\text{path}_2} T_0(\text{base})$
- $\widetilde{\text{surf}}$: $T_1(\text{path}_1) \circ' T_1(\text{path}_2) =^{T_0(\text{base}) =^C_{-} T_0(\text{base})}_{\text{surf}} T_1(\text{path}_2) \circ' T_1(\text{path}_1)$

Elimination of the torus

Four constructors of the torus T^2:

- $\mathrm{base} : T^2$
- $\mathrm{path}_1 : \mathrm{base} =_{T^2} \mathrm{base}$
- $\mathrm{path}_2 : \mathrm{base} =_{T^2} \mathrm{base}$
- $\mathrm{surf} : \mathrm{path}_1 \circ \mathrm{path}_2 =_{\mathrm{base} =_{T^2} \mathrm{base}} \mathrm{path}_2 \circ \mathrm{path}_1$

Thus:

- $\widetilde{\mathrm{base}} : C(\mathrm{base})$
- $\widetilde{\mathrm{path}_1} : \widetilde{\mathrm{base}} =^{C}_{\mathrm{path}_1} \widetilde{\mathrm{base}}$
- $\widetilde{\mathrm{path}_2} : \widetilde{\mathrm{base}} =^{C}_{\mathrm{path}_2} \widetilde{\mathrm{base}}$
- $\widetilde{\mathrm{surf}} : \widetilde{\mathrm{path}_1} \circ' \widetilde{\mathrm{path}_2} =^{\widetilde{\mathrm{base}} =^{C}_{-} \widetilde{\mathrm{base}}}_{\mathrm{surf}} \widetilde{\mathrm{path}_2} \circ' \widetilde{\mathrm{path}_1}$

Equality rules for the torus

Assume we have

- $\widetilde{\text{base}} : C(\text{base})$
- $\widetilde{\text{path}_1} : \widetilde{\text{base}} =^{C}_{\text{path}_1} \widetilde{\text{base}}$
- $\widetilde{\text{path}_2} : \widetilde{\text{base}} =^{C}_{\text{path}_2} \widetilde{\text{base}}$
- $\widetilde{\text{surf}} : \widetilde{\text{path}_1} \circ' \widetilde{\text{path}_2} =^{\widetilde{\text{base}} =^{C}_{-} \widetilde{\text{base}}}_{\text{surf}} \widetilde{\text{path}_2} \circ' \widetilde{\text{path}_1}$

Then there exists a function $f : (x : H) \to C(x)$ such that

- $f(\text{base}) = \widetilde{\text{base}}$
- $\mathbf{apd}_f(\text{path}_1) = \widetilde{\text{path}_1}$
- $\mathbf{apd}_f(\text{path}_2) = \widetilde{\text{path}_2}$
- $\mathbf{apd}^2_f(\text{surf}) = \widetilde{\text{surf}}$

Conclusion

- Extension from 1-Hit schema to 2-Hit schema
- Elimination rules for 2-Hits