

predicate logic & dependent types

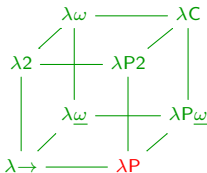
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Type Theory & Rocq

2025–2026

Radboud University Nijmegen

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introduction

function types become dependent

for predicate logic one needs to generalize

$$(\lambda x : A. M) : A \rightarrow B$$

function types

to

$$(\lambda x : A. M) : \Pi x : A. B$$

dependent function types

introduction

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can contain x



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dependent function types

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function types become dependent

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to

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$$(\lambda x : A. M) : \Pi x : A. B$$

dependent function types

$$\forall x : A. B$$

forall $x : A, B$

three different notations for the same type

we skip chapters 3 and 5 for now

inductive types



next week

program extraction



last week (briefly)

two weeks ago and this week

Curry-Howard correspondence:

propositional logic	$\lambda \rightarrow$	simple types
predicate logic	λP	dependent types
second order logic	$\lambda 2$	polymorphic types
beyond minimal logic	CIC	inductive types

two weeks ago and this week

Curry-Howard correspondence:

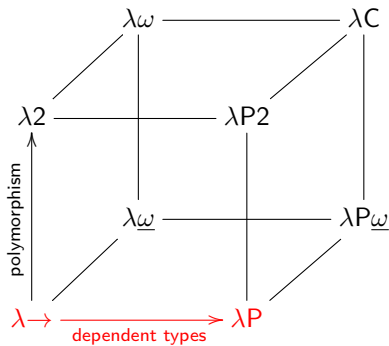
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dependent types in the lambda cube



recap: propositional logic

$$A, B ::= a \mid A \rightarrow B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \perp$$

$$\begin{array}{ll} I \rightarrow & E \rightarrow \\ I \wedge & El \wedge \quad Er \wedge \\ Il \vee \quad Ir \vee & E \vee \\ I \neg & E \neg \\ I \top & E \perp \end{array}$$

recap: simply typed lambda calculus

$$\begin{aligned} A, B &::= a \mid A \rightarrow B \\ M, N &::= x \mid MN \mid \lambda x : A. M \end{aligned}$$

$$\frac{}{\Gamma \vdash x : A} \quad \text{for } (x : A) \in \Gamma$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A. M) : A \rightarrow B}$$

$$\frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

predicate logic

three predicate logics

- ▶ minimal predicate logic

$$\rightarrow \forall$$

- ▶ constructive predicate logic

$$\rightarrow \wedge \vee \neg \top \perp \forall \exists$$

- ▶ classical predicate logic

$$\begin{aligned} &A \vee \neg A \\ &\neg\neg A \rightarrow A \end{aligned}$$

propositional versus predicate logic: syntax

$$A, B ::= a \mid A \rightarrow B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \perp$$

$$M, N ::= x \mid f(\vec{M})$$

$$\vec{M} ::= \cdot \mid \vec{M}, N$$

$$A, B ::= p(\vec{M}) \mid A \rightarrow B \mid A \wedge B \mid A \vee B \mid \neg A \mid \top \mid \perp \mid \\ \forall x. A \mid \exists x. A$$

$\forall$ and $\exists$ bind weakly

minimal propositional versus minimal predicate logic: rules

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ B \end{array}}{A \rightarrow B} I[H] \rightarrow$$

$$\frac{\begin{array}{c} \vdots \\ A \rightarrow B \end{array} \quad \begin{array}{c} \vdots \\ A \end{array}}{B} E \rightarrow$$

$$\frac{\begin{array}{c} \vdots \\ A \end{array}}{\forall x. A} I\forall$$

$$\frac{\begin{array}{c} \vdots \\ \forall x. A \end{array}}{A[x := M]} E\forall$$

variable condition of $I\forall$: x not free in available assumptions

same four rules with explicit assumptions

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} I \rightarrow$$

$$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} E \rightarrow$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x. A} I \forall$$

$$\frac{\Gamma \vdash \forall x. A}{\Gamma \vdash A[x := M]} E \forall$$

variable condition of $I \forall$: x not free in Γ

rules for the existential quantifier

$$\frac{\vdots \quad A[x := M]}{\exists x. A} I\exists \qquad \frac{\vdots \quad \exists x. A \quad \vdots \quad \forall x. A \rightarrow C}{C} E\exists$$

variable condition of $E\exists$: x not free in C

example proof in minimal logic

$$\forall x. ((\forall y. p(y)) \rightarrow p(x))$$

example proof in minimal logic

$$\forall x. ((\forall y. p(y)) \rightarrow p(x))$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ B \end{array}}{A \rightarrow B} I[H] \rightarrow \quad \frac{\begin{array}{c} \vdots \\ A \end{array}}{\forall x. A} I\forall \quad \frac{\begin{array}{c} \vdots \\ \forall x. A \end{array}}{A[x := M]} E\forall$$

example proof in minimal logic

$$\frac{(\forall y. p(y)) \rightarrow p(x)}{\forall x. ((\forall y. p(y)) \rightarrow p(x))} I\forall$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ B \end{array}}{A \rightarrow B} I[H] \rightarrow$$

$$\frac{\begin{array}{c} \vdots \\ \textcolor{red}{A} \end{array}}{\textcolor{green}{\forall x. A}} I\forall$$

$$\frac{\begin{array}{c} \vdots \\ \forall x. A \end{array}}{A[x := M]} E\forall$$

example proof in minimal logic

$$[\forall y. p(y)^H]$$

$$\frac{\frac{p(x)}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. ((\forall y. p(y)) \rightarrow p(x))} I\forall$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ \textcolor{red}{B} \end{array}}{\textcolor{green}{A} \rightarrow \textcolor{green}{B}} I[H] \rightarrow \quad \frac{\begin{array}{c} \vdots \\ A \end{array}}{\forall x. A} I\forall \quad \frac{\begin{array}{c} \vdots \\ \forall x. A \end{array}}{A[x := M]} E\forall$$

example proof in minimal logic

$[\forall y. p(y)^H]$

$$\frac{\frac{\frac{\forall y. p(y)}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. ((\forall y. p(y)) \rightarrow p(x))} I\forall$$

$$\frac{\frac{\vdots}{B} I[H] \rightarrow}{A \rightarrow B} \quad \frac{\frac{\vdots}{A} I\forall}{\forall x. A} \quad \frac{\frac{\vdots}{\forall x. A} E\forall}{A[x := M]}$$

example proof in minimal logic

$[\forall y. p(y)^H]$

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. ((\forall y. p(y)) \rightarrow p(x))} I\forall$$

$$\begin{array}{ccc} \frac{\frac{[A^H]}{\vdots} B}{A \rightarrow B} I[H] \rightarrow & \frac{\frac{\vdots}{A}}{\forall x. A} I\forall & \frac{\frac{\vdots}{\forall x. A}}{A[x := M]} E\forall \end{array}$$

example proof in minimal logic

$[\forall y. p(y)^H]$

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. ((\forall y. p(y)) \rightarrow p(x))} I\forall$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions
(there are no available assumptions at that point)

example proof in minimal logic

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. ((\forall y. p(y)) \rightarrow p(x))} I\forall$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions
(there are no available assumptions at that point)

example with existential quantifier

$$(\exists x. p(x)) \leftrightarrow \neg(\forall x. \neg p(x))$$

example with existential quantifier

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

$$(\exists x. p(x)) \leftrightarrow \neg(\forall x. \neg p(x))$$

$$\begin{aligned} &(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x)) \\ &\neg(\forall x. \neg p(x)) \rightarrow (\exists x. p(x)) \end{aligned}$$

example with existential quantifier

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

$$(\exists x. p(x)) \leftrightarrow \neg(\forall x. \neg p(x))$$

not constructively valid

$$\begin{aligned} &(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x)) \\ &\neg(\forall x. \neg p(x)) \rightarrow (\exists x. p(x)) \end{aligned}$$

example with existential quantifier

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

$$(\exists x. p(x)) \leftrightarrow \neg(\forall x. \neg p(x))$$

not constructively valid

$$\begin{aligned} &(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x)) \\ &\neg(\forall x. \neg p(x)) \rightarrow \textcolor{blue}{\neg}(\exists x. p(x)) \end{aligned}$$

example from left to right

$$(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))$$

example from left to right

$$(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ B \\ \hline A \rightarrow B \end{array} I[H] \rightarrow}{\quad} \quad \frac{\begin{array}{c} [A^H] \\ \vdots \\ \perp \\ \hline \neg A \end{array} I[H] \neg}{\quad} \quad \frac{\begin{array}{c} \vdots \\ \exists x. A \end{array} \quad \begin{array}{c} \vdots \\ \forall x. A \rightarrow C \\ \hline C \end{array} E\exists}{\quad}$$

example from left to right

$$[\exists x. p(x)]^{H_0}$$

$$\frac{\neg(\forall x. \neg p(x))}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ \textcolor{red}{B} \end{array}}{\textcolor{green}{A} \rightarrow \textcolor{green}{B}} I[H] \rightarrow$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ \perp \end{array}}{\neg A} I[H] \neg$$

$$\frac{\begin{array}{c} \vdots \\ \exists x. A \end{array} \quad \begin{array}{c} \vdots \\ \forall x. A \rightarrow C \end{array}}{C} E\exists$$

example from left to right

$$[\exists x. p(x)^{H_0}] \quad [\forall x. \neg p(x)^{H_1}]$$

$$\frac{\frac{\frac{\perp}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow}{\quad}$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ B \end{array}}{A \rightarrow B} I[H] \rightarrow$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ \perp \\ \neg A \end{array}}{I[H] \neg}$$

$$\frac{\begin{array}{c} \vdots \\ \exists x. A \end{array} \quad \begin{array}{c} \vdots \\ \forall x. A \rightarrow C \end{array}}{C} E\exists$$

example from left to right

$$[\exists x. p(x)^{H_0}] \quad [\forall x. \neg p(x)^{H_1}]$$

$$\frac{\frac{\frac{\exists x. p(x)}{\perp} \quad \forall x. p(x) \rightarrow \perp}{E\exists}}{\frac{\frac{\perp}{\neg(\forall x. \neg p(x))} \quad I[H_1]\neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0]\rightarrow}$$

$$\frac{\frac{[A^H] \quad \vdots \quad B}{A \rightarrow B} I[H]\rightarrow}{\frac{[A^H] \quad \vdots \quad \perp}{\neg A} I[H]\neg} \quad \frac{\frac{\vdots \quad \exists x. A}{\forall x. A \rightarrow C} E\exists}{\vdots \quad C}$$

example from left to right

$$[\exists x. p(x)^{H_0}] \quad [\forall x. \neg p(x)^{H_1}]$$

$$\frac{[\exists x. p(x)^{H_0}] \quad \forall x. p(x) \rightarrow \perp}{E\exists}$$

$$\frac{\perp}{\neg(\forall x. \neg p(x))} I[H_1]\neg$$

$$\frac{\neg(\forall x. \neg p(x))}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0]\rightarrow$$

$$\frac{\begin{array}{c} [A^H] \\ \vdots \\ B \end{array}}{A \rightarrow B} I[H]\rightarrow \quad \frac{\begin{array}{c} [A^H] \\ \vdots \\ \perp \end{array}}{\neg A} I[H]\neg \quad \frac{\begin{array}{c} \vdots \\ \exists x. A \end{array} \quad \begin{array}{c} \vdots \\ \forall x. A \rightarrow C \end{array}}{C} E\exists$$

example from left to right

$$[\exists x. p(x)^{H_0}] \quad [\forall x. \neg p(x)^{H_1}]$$

$$\frac{[\exists x. p(x)^{H_0}] \quad [\forall x. p(x) \rightarrow \perp^{H_1}]}{E\exists}$$

$$\frac{\perp}{\neg(\forall x. \neg p(x))} I[H_1]\neg$$

$$\frac{\neg(\forall x. \neg p(x))}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0]\rightarrow$$

$$\frac{[A^H] \quad \vdots \quad B}{A \rightarrow B} I[H]\rightarrow$$

$$\frac{[A^H] \quad \vdots \quad \perp}{\neg A} I[H]\neg$$

$$\frac{\vdots \quad \exists x. A \quad \vdots \quad \forall x. A \rightarrow C}{C} E\exists$$

example from left to right

$$\frac{\frac{\frac{[\exists x. p(x)^{H_0}] \quad [\forall x. p(x) \rightarrow \bot^{H_1}]}{E\exists} \quad \bot}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

variable condition check:

at the $E\exists$ step the variable x does not occur in $\bot$

example from left to right

$$\frac{\frac{\frac{[\exists x. p(x)^{H_0}] \quad [\forall x. p(x) \rightarrow \perp^{H_1}]}{E\exists} \quad \perp}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

variable condition check:

at the $E\exists$ step the variable x does not occur in $\perp$

example from right to left

$$\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}]$$

$$\frac{\neg\neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0] \rightarrow$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}]$$

$$\frac{\frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}]$$

$$\frac{\frac{\frac{\neg(\forall x. \neg p(x)) \quad \forall x. \neg p(x)}{E_{\neg}}}{\frac{\perp}{I[H_1]_{\neg}}}{\neg(\exists x. p(x))} \quad I[H_0]_{\rightarrow}$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}]$$

$$\frac{\frac{\frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \forall x. \neg p(x)}{E\neg} \quad \perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}]$$

$$\frac{\frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \frac{\neg p(x)}{\forall x. \neg p(x)} I\forall}{E\neg} \quad \perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg \quad \frac{\neg\neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}] \quad [p(x)^{H_2}]$$

$$\frac{\frac{\frac{\perp}{I[H_2]\neg} \quad \frac{\neg p(x)}{I\forall} \quad \forall x. \neg p(x)}{E\neg} \quad [\neg(\forall x. \neg p(x))^{H_0}]}{\frac{\perp}{I[H_1]\neg} \quad \neg\neg(\exists x. p(x))} \quad \neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x)) \quad I[H_0]\rightarrow$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}] \quad [p(x)^{H_2}]$$

$$\begin{array}{c}
 \frac{\neg(\exists x. p(x)) \quad \exists x. p(x)}{E \rightarrow} \\
 \frac{\frac{\frac{\perp}{\neg p(x)} I[H_2] \neg}{\forall x. \neg p(x)} I \forall}{\frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \forall x. \neg p(x)}{E \neg}} \\
 \frac{\perp}{\frac{\neg \neg(\exists x. p(x))}{I[H_1] \neg}} \\
 \frac{\neg \neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg \neg(\exists x. p(x))} I[H_0] \rightarrow
 \end{array}$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}] \quad [p(x)^{H_2}]$$

$$\begin{array}{c}
 \dfrac{[\neg(\exists x. p(x))^{H_1}] \quad \exists x. p(x)}{E \rightarrow} \\
 \dfrac{\perp}{I[H_2] \neg} \\
 \dfrac{\neg p(x)}{I \forall} \\
 \dfrac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \forall x. \neg p(x)}{E \neg} \\
 \dfrac{\perp}{I[H_1] \neg} \\
 \dfrac{\neg \neg(\exists x. p(x))}{I[H_0] \rightarrow}
 \end{array}$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}] \quad [p(x)^{H_2}]$$

$$\begin{array}{c}
 \frac{[\neg(\exists x. p(x))^{H_1}] \quad \frac{p(x)}{\exists x. p(x)} I\exists}{E\rightarrow} \\
 \frac{\perp}{\neg p(x)} I[H_2]\neg \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \forall x. \neg p(x)}{E\neg} I\forall \\
 \frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg \\
 \frac{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))}{I[H_0]\rightarrow}
 \end{array}$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}] \quad [p(x)^{H_2}]$$

$$\begin{array}{c}
 \dfrac{\dfrac{[\neg(\exists x. p(x))^{H_1}] \quad \dfrac{[p(x)^{H_2}]}{\exists x. p(x)} I\exists}{\perp} E\rightarrow}{\dfrac{\perp}{\neg p(x)} I[H_2]\neg} \\
 \dfrac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \dfrac{\forall x. \neg p(x)}{I\forall}}{E\neg} \\
 \dfrac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg \\
 \dfrac{\neg\neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow
 \end{array}$$

example from right to left

$$[\neg(\forall x. \neg p(x))^{H_0}] \quad [\neg(\exists x. p(x))^{H_1}] \quad [p(x)^{H_2}]$$

$$\begin{array}{c}
 \frac{[\neg(\exists x. p(x))^{H_1}] \quad \frac{[p(x)^{H_2}]}{\exists x. p(x)} I\exists}{E\rightarrow} \\
 \frac{\perp}{\neg p(x)} I[H_2]\neg \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \frac{\forall \textcolor{teal}{x}. \neg p(x)}{I\forall} \textcolor{red}{I\forall}}{E\neg} \\
 \frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg \\
 \frac{\neg\neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow
 \end{array}$$

variable condition check:

at the $I\forall$ step the variable $\textcolor{teal}{x}$ does not occur in assumptions

example from right to left

$$\begin{array}{c}
 \frac{\frac{[\neg(\exists x. p(x))^{H_1}] \quad \frac{[p(x)^{H_2}]}{\exists x. p(x)} I\exists}{E\rightarrow} \quad \frac{\frac{\perp}{\neg p(x)} I[H_2]\neg}{\forall x. \neg p(x)} I\forall}{E\neg} \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg}{I[H_0]\rightarrow}
 \end{array}$$

variable condition check:

at the $I\forall$ step the variable x does not occur in assumptions

dependent type theory

types thus far

- ▶ atomic types
 $a, b, c, \dots$
- ▶ types of proof objects of propositions
 $a, b, c, \dots$
- ▶ function types
- ▶ **datatypes**
 - ▶ natural numbers
 - ▶ Booleans
 - ▶ lists
 - ▶ binary trees
 - ▶ $\dots$

how are types introduced?

- ▶ free type variables

STT = simple type theory

how are types introduced?

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STT = simple type theory

- ▶ in the context

PTSs = pure type systems: $\lambda \rightarrow$ λP $\lambda 2$ λC

how are types introduced?

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PTSs = pure type systems: $\lambda \rightarrow$ λP $\lambda 2$ λC

$$\Gamma \vdash M : A$$

$$O : \text{nat}, S : \text{nat} \rightarrow \text{nat} \vdash S (S (S O)) : \text{nat}$$

how are types introduced?

- ▶ free type variables

STT = simple type theory

- ▶ in the context

PTSs = pure type systems: $\lambda \rightarrow$ λP $\lambda 2$ λC

$$\Gamma \vdash M : A$$

$\text{nat} : *$, $O : \text{nat}$, $S : \text{nat} \rightarrow \text{nat} \vdash S (S (S O)) : \text{nat}$

how are types introduced?

- ▶ free type variables

STT = simple type theory

- ▶ in the context = 'axiomatic'

PTSs = pure type systems: $\lambda \rightarrow$ λP $\lambda 2$ λC

$$\Gamma \vdash M : A$$

$\text{nat} : *$, $O : \text{nat}$, $S : \text{nat} \rightarrow \text{nat} \vdash S (S (S O)) : \text{nat}$

Rocq: Parameter

how are types introduced?

- ▶ free type variables

STT = simple type theory

- ▶ in the context = 'axiomatic'

PTSs = pure type systems: $\lambda \rightarrow \lambda P \ \lambda 2 \ \lambda C$

$$\Gamma \vdash M : A$$

$$\text{nat} : *, O : \text{nat}, S : \text{nat} \rightarrow \text{nat} \vdash S (S (S O)) : \text{nat}$$

Rocq: Parameter

- ▶ definitions $\longrightarrow$ next week

CIC = Calculus of Inductive Constructions

$$E[\Gamma] \vdash M : A$$

Rocq: Definition
Inductive

star and box

the **sorts** of the lambda cube:

***** = the type of types

□ = the type of *

star and box

the **sorts** of the lambda cube:

***** = the type of types

□ = the type of *

the sort * is a **kind**
= third of four levels

object : type

type : kind

kind : □

object : type : **kind** : □

S : nat → nat : ***** : □

example contexts

natural numbers:

$\text{nat} : *$

$0 : \text{nat}$

$S : \text{nat} \rightarrow \text{nat}$

$\text{add} : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$

Booleans:

$\text{bool} : *$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

lists

lists of Booleans:

```
list : *  
nil : list  
cons : bool → list → list  
hd : list → bool  
tl : list → list  
append : list → list → list  
reverse : list → list
```

lists

lists of Booleans:

```
list : *  
nil : list  
cons : bool → list → list  
hd : list → bool  
tl : list → list  
append : list → list → list  
reverse : list → list
```

cons has two arguments
(the head and tail to be put together)

trueslist

the function 'trueslist' maps a number n to a list of length n consisting of n copies of 'true'

$$\text{trueslist } 0 = \text{nil}$$
$$\text{trueslist } (S\ 0) = \text{cons true nil}$$
$$\text{trueslist } (S\ (S\ 0)) = \text{cons true (cons true nil)}$$
$$\text{trueslist } (S\ (S\ (S\ 0))) = \text{cons true (cons true (cons true nil))}$$
$$\dots$$
$$\text{trueslist} : \text{nat} \rightarrow \text{list}$$

vectors and truesvec

vectors of Booleans:

$\text{vec} : \text{nat} \rightarrow *$

$\text{vec } 0 : *$ vectors of length 0

$\text{vec } (S\ 0) : *$ vectors of length 1

$\text{vec } (S\ (S\ 0)) : *$ vectors of length 2

...

vectors and truesvec

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$\text{vec} : \text{nat} \rightarrow *$

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...

$\text{trueslist} : \text{nat} \rightarrow \text{list}$

$\text{truesvec} : \text{nat} \rightarrow \text{vec}$

vectors and truesvec

vectors of Booleans:

$\text{vec} : \text{nat} \rightarrow *$	
$\text{vec } 0 : *$	vectors of length 0
$\text{vec } (S\ 0) : *$	vectors of length 1
$\text{vec } (S\ (S\ 0)) : *$	vectors of length 2
$\dots$	

$\text{trueslist} : \text{nat} \rightarrow \text{list}$

$\text{truesvec} : \text{nat} \rightarrow \text{vec } n$

vectors and truesvec

vectors of Booleans:

$\text{vec} : \text{nat} \rightarrow *$	
$\text{vec } 0 : *$	vectors of length 0
$\text{vec } (S\ 0) : *$	vectors of length 1
$\text{vec } (S\ (S\ 0)) : *$	vectors of length 2
$\dots$	

$\text{trueslist} : \text{nat} \rightarrow \text{list}$

$\text{truesvec} : \text{nat} \rightarrow \text{vec } n$

$\text{truesvec} : \prod n : \text{nat}. \text{vec } n$

vnil and vcons

$\text{vnil} : \text{vec } 0$

$\text{vcons} : \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (\text{S } n)$

vnil and vcons

$\text{vnil} : \text{vec } 0$

$\text{vcons} : \text{nat} \rightarrow \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (\text{S } n)$

vnil and vcons

$\text{vnil} : \text{vec } 0$

$\text{vcons} : \text{nat} \rightarrow \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

$\text{vcons} : \Pi n : \text{nat}. \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

vnil and vcons

$\text{vnil} : \text{vec } 0$

$\text{vcons} : \text{nat} \rightarrow \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

$\text{vcons} : \Pi n : \text{nat}. \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

vcons has three arguments!

vnil and vcons

$\text{vnil} : \text{vec } 0$

$\text{vcons} : \text{nat} \rightarrow \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

$\text{vcons} : \Pi n : \text{nat}. \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

vcons has three arguments!

$\text{truesvec } (S\ (S\ 0)) : \text{vec } (S\ (S\ 0))$

$\text{truesvec } (S\ (S\ 0)) = \text{vcons } (S\ 0)\ \text{true}\ (\text{vcons } 0\ \text{true}\ \text{vnil})$

types when applying vcons

$\text{truesvec } (S\ O) = \text{vcons } O\ \text{true}\ \text{vnil}$

$\text{truesvec} : \prod n : \text{nat}. \text{vec } n$

$\text{truesvec } (S\ O) : \text{vec } (S\ O)$

$\text{vcons} : \prod n : \text{nat}. \text{bool} \rightarrow \text{vec } n \rightarrow \text{vec } (S\ n)$

$O : \text{nat}$

$\text{vcons } O : \text{bool} \rightarrow \text{vec } O \rightarrow \text{vec } (S\ O)$

$\text{true} : \text{bool}$

$\text{vcons } O\ \text{true} : \text{vec } O \rightarrow \text{vec } (S\ O)$

$\text{vnil} : \text{vec } O$

$\text{vcons } O\ \text{true}\ \text{vnil} : \text{vec } (S\ O)$

BHK-interpretation

proof of $A \rightarrow B$

proof of $\neg A$

proof of $A \wedge B$

proof of $A \vee B$

function from proofs of A to proofs of B

function from proofs of A to proofs of $\perp$

pair of a proof of A and a proof of B

either a proof of A or a proof of B

BHK-interpretation

proof of $A \rightarrow B$	function from proofs of A to proofs of B
proof of $\neg A$	function from proofs of A to proofs of $\perp$
proof of $A \wedge B$	pair of a proof of A and a proof of B
proof of $A \vee B$	either a proof of A or a proof of B
proof of $\forall x. A$	dependent function from objects to proofs of A
proof of $\exists x. A$	dependent pair of an object and a proof of A

dependent = A is not fixed but *depends* on the object x

BHK-interpretation

proof of $A \rightarrow B$	function from proofs of A to proofs of B
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proof of $\forall x. A$	dependent function from objects to proofs of A
proof of $\exists x. A$	dependent pair of an object and a proof of A

dependent = A is not fixed but *depends* on the object x

$$A \vee B \leftrightarrow \exists x \in \{\text{left}, \text{right}\}. (x = \text{left} \rightarrow A) \wedge (x = \text{right} \rightarrow B)$$

Curry-Howard correspondence

type theory

$A \rightarrow B$

$\Pi x : D. A$

Rocq

$A \multimap B$

`forall x : D, A`

minimal logic

$A \rightarrow B$

$\forall x. A$

Curry-Howard correspondence

type theory

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$$A \rightarrow B$$

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Rocq

$$A \rightarrow B$$

$$\text{forall } x : D, A$$

minimal logic

$$A \rightarrow B$$

$$\forall x. A$$

$D : *$ is called **Terms** in the notes

Curry-Howard correspondence

type theory	Rocq	minimal logic
$A \rightarrow B$	$A \rightarrow B$	$A \rightarrow B$
$\Pi x : D. A$	<code>forall x : D, A</code>	$\forall x. A$

$D : *$ is called `Terms` in the notes

proof term	type	statement proved
H	$: A \rightarrow B$	$A \rightarrow B$
H'	$: A$	A
HH'	$: B$	B

Curry-Howard correspondence

type theory	Rocq	minimal logic
$A \rightarrow B$	$A \rightarrow B$	$A \rightarrow B$
$\Pi x : D. A$	<code>forall x : D, A</code>	$\forall x. A$

$D : *$ is called `Terms` in the notes

proof term	type	statement proved
H	$: A \rightarrow B$	$A \rightarrow B$
H'	$: A$	A
HH'	$: B$	B
H	$: \Pi x : D. Px$	$\forall x. P(x)$
HM	$: PM$	$P(M)$

Rocq version of the first example

Parameter D : Set.

D is declared

Rocq version of the first example

Parameter D : Set.

p is declared

Parameter p : D -> Prop.

Rocq version of the first example

Parameter D : Set.

1 subgoal (ID 1)

Parameter p : D -> Prop.

Lemma four :

forall x : D,
 (forall y : D, p y) ->
 p x.

=====
forall x : D,
 (forall y : D, p y) -> p x

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

Rocq version of the first example

Parameter D : Set.

1 subgoal (ID 3)

Parameter p : D -> Prop.

Lemma four :

forall x : D,
 (forall y : D, p y) ->
 p x.

intros x H.

x : D

H : forall y : D, p y

=====

p x

$$\frac{\frac{p(x)}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. (\forall y. p(y)) \rightarrow p(x)} I\forall$$

Rocq version of the first example

Parameter D : Set.

No more subgoals.

Parameter p : D -> Prop.

Lemma four :

forall x : D,

(forall y : D, p y) ->

p x.

intros x H.

apply H.

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. (\forall y. p(y)) \rightarrow p(x)} I\forall$$

Rocq version of the first example

Parameter D : Set.

Parameter p : D -> Prop.

Lemma four :

forall x : D,
 (forall y : D, p y) ->
 p x.

intros x H.

apply H.

Qed.

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. (\forall y. p(y)) \rightarrow p(x)} I\forall$$

Rocq version of the first example

```
Parameter D : Set.  
Parameter p : D -> Prop.
```

```
Lemma four :  
  forall x : D,  
    (forall y : D, p y) ->  
      p x.  
intros x H.  
apply H.  
Qed.
```

Print four.

```
four =  
fun (x : D) (H : forall y : D, p y)  
=> H x  
      : forall x : D,  
        (forall y : D, p y) -> p x  
Arguments four _ _%function_scope
```

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. (\forall y. p(y)) \rightarrow p(x)} I\forall$$

$$(\lambda x : D. \lambda H : (\Pi y : D. py). Hx) : \Pi x : D. (\Pi y : D. py) \rightarrow px$$

Rocq version of the first example

```
Parameter D : Set.  
Parameter p : D -> Prop.
```

```
Lemma four :  
  forall x : D,  
    (forall y : D, p y) ->  
      p x.  
intros x H.  
apply H.  
Qed.
```

```
Print four.
```

```
four =  
fun (x : D) (H : forall y : D, p y)  
=> H x
```

```
: forall x : D,  
  (forall y : D, p y) -> p x
```

```
Arguments four _ _%function_scope
```

$$\frac{\frac{\frac{[\forall y. p(y)^H]}{p(x)} E\forall}{(\forall y. p(y)) \rightarrow p(x)} I[H] \rightarrow}{\forall x. (\forall y. p(y)) \rightarrow p(x)} I\forall$$

$$(\lambda x : D. \lambda H : (\Pi y : D. py). Hx) : \Pi x : D. (\Pi y : D. py) \rightarrow px$$

understanding the proof term

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

understanding the proof term

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

$$\lambda x : D.$$

understanding the proof term

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

$$\lambda x : D. \lambda H : \underbrace{(\Pi y : D. py)}_{\forall y. p(y)}.$$

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

$$\lambda x : D. \lambda H : (\Pi y : D. py). \underbrace{Hx}_{: px}$$

$$\forall x. (\forall y. p(y)) \rightarrow p(x)$$

$$\lambda x : D. \lambda H : (\Pi y : D. py). Hx$$

the second example in Rocq, from left to right

Parameter D : Set.

1 subgoal (ID 3)

Parameter p : D -> Prop.

Lemma five :

(exists x : D, p x) ->
~ (forall x : D, ~ p x).

=====

(exists x : D, p x) ->
~ (forall x : D, ~ p x)

$$(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))$$

the second example in Rocq, from left to right

```
Parameter D : Set.                                1 subgoal (ID 6)
Parameter p : D -> Prop.
Lemma five :
  (exists x : D, p x) ->
  ~ (forall x : D, ~ p x).
intros H0 H1.
```

H0 : exists x : D, p x
H1 : forall x : D, ~ p x
=====

False

$$\frac{\frac{\perp}{\neg(\forall x. \neg p(x))} \textcolor{red}{I[H_1]}\neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} \textcolor{red}{I[H_0]}\rightarrow$$

the second example in Rocq, from left to right

```
Parameter D : Set.
Parameter p : D -> Prop.

Lemma five :
  (exists x : D, p x) ->
    ~ (forall x : D, ~ p x).
intros H0 H1.
elim H0.
```

1 subgoal (ID 7)

H0 : exists x : D, p x
H1 : forall x : D, ~ p x
=====

forall x : D, p x -> False

$$\frac{\frac{\frac{[\exists x. p(x)]^{H_0} \quad \forall x. p(x) \rightarrow \perp}{\perp} E\exists}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

the second example in Rocq, from left to right

Parameter D : Set.

No more subgoals.

Parameter p : D -> Prop.

Lemma five :

(exists x : D, p x) ->
~ (forall x : D, ~ p x).

intros H0 H1.

elim H0.

apply H1.

$$\frac{\frac{\frac{[\exists x. p(x)]^{H_0} \quad [\forall x. p(x) \rightarrow \perp]^{H_1}}{\perp} E\exists}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

the second example in Rocq, from left to right

Parameter D : Set.

Parameter p : D -> Prop.

Lemma five :

(exists x : D, p x) ->
~ (forall x : D, ~ p x).

intros H0 H1.

elim H0.

apply H1.

Qed.

$$\frac{\frac{\frac{[\exists x. p(x)^{H_0}] \quad [\forall x. p(x) \rightarrow \perp^{H_1}]}{E\exists}}{\perp}}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

the second example in Rocq, from left to right

Parameter D : Set.

Parameter p : D -> Prop.

Lemma five :

(exists x : D, p x) ->
~ (forall x : D, ~ p x).

intros H0 H1.

elim H0.

apply H1.

Qed.

$$\frac{\frac{\frac{[\exists x. p(x)^{H_0}] \quad [\forall x. p(x) \rightarrow \perp^{H_1}]}{\perp} E\exists}{\neg(\forall x. \neg p(x))} I[H_1] \neg}{(\exists x. p(x)) \rightarrow \neg(\forall x. \neg p(x))} I[H_0] \rightarrow$$

the second example in Rocq, from right to left

Parameter D : Set.

1 subgoal (ID 5)

Parameter p : D -> Prop.

Lemma six :

~ (forall x : D, ~ p x) ->
~ ~ (exists x : D, p x).

=====

~ (forall x : D, ~ p x) ->
~ ~ (exists x : D, p x)

$$\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))$$

the second example in Rocq, from right to left

```
Parameter D : Set.                1 subgoal (ID 8)
Parameter p : D -> Prop.

Lemma six :                       H0 : ~ (forall x : D, ~ p x)
  ~ (forall x : D, ~ p x) ->      H1 : ~ (exists x : D, p x)
  ~ ~ (exists x : D, p x).        =====
                                   False

intros H0 H1.
```

$$\frac{\frac{\perp}{\neg\neg(\exists x. p(x))} \quad I[H_1]\neg}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} \quad I[H_0]\rightarrow$$

the second example in Rocq, from right to left

```

Parameter D : Set.
Parameter p : D -> Prop.

1 subgoal (ID 9)

H0 : ~ (forall x : D, ~ p x)
H1 : ~ (exists x : D, p x)
=====
~ (forall x : D, ~ p x) ->
~ ~ (exists x : D, p x).
forall x : D, ~ p x

Lemma six :
  intros H0 H1.
  apply H0.

```

$$\frac{
 \frac{
 \frac{
 [\neg(\forall x. \neg p(x))^{H_0}] \quad \forall x. \neg p(x)
 }{
 \perp
 } E_{\neg}
 }{
 \neg\neg(\exists x. p(x))
 } I[H_1]\neg
 }{
 \neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))
 } I[H_0]\rightarrow$$

the second example in Rocq, from right to left

```

Parameter D : Set.
Parameter p : D -> Prop.

1 subgoal (ID 12)

H0 : ~ (forall x : D, ~ p x)
H1 : ~ (exists x : D, p x)
x : D
H2 : p x
=====
False

Lemma six :
  ~ (forall x : D, ~ p x) ->
  ~ ~ (exists x : D, p x).
intros H0 H1.
apply H0.
intros x H2.

```

$$\frac{
 \frac{
 \frac{
 \frac{
 \frac{\perp}{\neg p(x)} I[H_2] \neg
 }{\forall x. \neg p(x)} I\forall
 }{[\neg(\forall x. \neg p(x))]^{H_0}] } E\neg
 }{\perp}
 }{\neg\neg(\exists x. p(x))} I[H_1] \neg
 }{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0] \rightarrow$$

the second example in Rocq, from right to left

Parameter D : Set.

1 subgoal (ID 13)

Parameter p : D -> Prop.

Lemma six :

~ (forall x : D, ~ p x) ->

~ ~ (exists x : D, p x).

intros H0 H1.

apply H0.

intros x H2.

apply H1.

H0 : ~ (forall x : D, ~ p x)

H1 : ~ (exists x : D, p x)

x : D

H2 : p x

exists x0 : D, p x0

$$\begin{array}{c}
 \frac{[\neg(\exists x. p(x))^{H_1}] \quad \exists x. p(x)}{E \rightarrow} \\
 \frac{\perp}{\neg p(x)} I[H_2] \neg \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \forall x. \neg p(x)}{E \neg} I\forall \\
 \frac{\perp}{\neg \neg(\exists x. p(x))} I[H_1] \neg \\
 \frac{\neg(\forall x. \neg p(x)) \rightarrow \neg \neg(\exists x. p(x))}{I[H_0] \rightarrow}
 \end{array}$$

the second example in Rocq, from right to left

Parameter D : Set.

1 subgoal (ID 15)

Parameter p : D -> Prop.

Lemma six :

~ (forall x : D, ~ p x) ->

~ ~ (exists x : D, p x).

intros H0 H1.

apply H0.

intros x H2.

apply H1.

exists x.

H0 : ~ (forall x : D, ~ p x)

H1 : ~ (exists x : D, p x)

x : D

H2 : p x

=====

p x

$$\begin{array}{c}
 \frac{[\neg(\exists x. p(x))^{H_1}] \quad \frac{p(x)}{\exists x. p(x)} I\exists}{\exists x. p(x)} E\rightarrow \\
 \frac{\perp}{\neg p(x)} I[H_2]\neg \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \frac{\neg p(x)}{\forall x. \neg p(x)} I\forall}{\forall x. \neg p(x)} E\neg \\
 \frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg \\
 \frac{\neg\neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow
 \end{array}$$

the second example in Rocq, from right to left

Parameter D : Set.

No more subgoals.

Parameter p : D -> Prop.

Lemma six :

~ (forall x : D, ~ p x) ->

~ ~ (exists x : D, p x).

intros H0 H1.

apply H0.

intros x H2.

apply H1.

exists x.

apply H2.

$$\begin{array}{c}
 \frac{[\neg(\exists x. p(x))^{H_1}] \quad \frac{[p(x)^{H_2}]}{\exists x. p(x)} I\exists}{\exists x. p(x)} E\rightarrow \\
 \frac{\perp}{\neg p(x)} I[H_2]\neg \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}] \quad \frac{\neg p(x)}{\forall x. \neg p(x)} I\forall}{\forall x. \neg p(x)} E\neg \\
 \frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg \\
 \frac{\neg\neg(\exists x. p(x))}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow
 \end{array}$$

the second example in Rocq, from right to left

Parameter D : Set.

Parameter p : D -> Prop.

Lemma six :

~ (forall x : D, ~ p x) ->

~ ~ (exists x : D, p x).

intros H0 H1.

apply H0.

intros x H2.

apply H1.

exists x.

apply H2.

Qed.

$$\begin{array}{c}
 \frac{\frac{[\neg(\exists x. p(x))^{H_1}]}{\frac{[p(x)^{H_2}]}{\exists x. p(x)} I\exists} E\rightarrow \\
 \frac{\frac{\perp}{\neg p(x)} I[H_2]\neg}{\forall x. \neg p(x)} I\forall \\
 \frac{[\neg(\forall x. \neg p(x))^{H_0}]}{E\neg} \\
 \frac{\frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1]\neg}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0]\rightarrow
 \end{array}$$

the second example in Rocq, from right to left

Parameter D : Set.

Parameter p : D -> Prop.

Lemma six :

~ (forall x : D, ~ p x) ->

~ ~ (exists x : D, p x).

intros H0 H1.

apply H0.

intros x H2.

apply H1.

exists x.

apply H2.

Qed.

$$\frac{\frac{[\neg(\exists x. p(x))^{H_1}]}{\frac{\frac{\perp}{\neg p(x)} I[H_2] \neg}{\forall x. \neg p(x)} I\forall}{\exists x. p(x)} I\exists}{E\rightarrow}$$
$$\frac{[\neg(\forall x. \neg p(x))^{H_0}]}{\frac{\frac{\perp}{\neg\neg(\exists x. p(x))} I[H_1] \neg}{\neg(\forall x. \neg p(x)) \rightarrow \neg\neg(\exists x. p(x))} I[H_0] \rightarrow}$$

proof rules versus Rocq tactics

$I \rightarrow I\forall$	intro intros
$E \rightarrow E\forall$	apply
$E\wedge E\vee E\exists$	elim destruct intro-patterns
$I\wedge$	split
$I\vee$	left right
$I\exists$	exists

detours

proof normalization

detour = **introduction** rule directly followed by a **elimination** rule for the same connective

detours

proof normalization

detour = **introduction** rule directly followed by a **elimination** rule for the same connective

cut = corresponding notion in sequent calculus

can be **eliminated** $\longrightarrow$ reduction of the proof term

detours

proof normalization

detour = **introduction** rule directly followed by a **elimination** rule for the same connective

cut = corresponding notion in sequent calculus

can be **eliminated** $\longrightarrow$ reduction of the proof term

detour for implication:

‘prove a **lemma** A and then prove B using this lemma’

detour elimination is inlining the proof₁ of the lemma everywhere

$$\frac{\frac{\frac{[A^H]}{\vdots_2} B}{A \rightarrow B} \textcolor{red}{I[H] \rightarrow} \quad \frac{\vdots_1}{A} \textcolor{red}{E \rightarrow}}{B} \longrightarrow_{\beta} \frac{\vdots_1}{A} \textcolor{red}{E \rightarrow} \quad \frac{\vdots_2}{B}$$

detours for predicate logic

detour for the universal quantifier

generalize the statement $A[x := M]$ to A with arbitrary x
elimination is specializing the proof₁ to M

$$\frac{\frac{\frac{\vdots_1}{A}}{\forall x. A} I\forall}{A[x := M]} E\forall \quad \longrightarrow_{\beta} \quad \frac{\vdots_1[x := M]}{A[x := M]}$$

$$(\lambda x : D. H_1)M \rightarrow_{\beta} H_1[x := M]$$

STT versus λP syntax

STT:

$$\begin{aligned}
A, B &::= a \mid A \rightarrow B \\
M, N &::= x \mid MN \mid \lambda x : A. M \\
\Gamma &::= \cdot \mid \Gamma, x : A \\
\mathcal{J} &::= \Gamma \vdash M : A
\end{aligned}$$

 λP :

$$\begin{aligned}
s &::= * \mid \square \\
M, N, A, B &::= x \mid MN \mid \lambda x : A. M \mid \overline{\Pi x : A. B} \mid s \\
\Gamma &::= \cdot \mid \Gamma, x : A \\
\mathcal{J} &::= \Gamma \vdash M : A
\end{aligned}$$

STT versus λP syntax

STT:

$$\begin{aligned}
A, B &::= a \mid A \rightarrow B \\
M, N &::= x \mid MN \mid \lambda x : A. M \\
\Gamma &::= \cdot \mid \Gamma, x : A \\
\mathcal{J} &::= \Gamma \vdash M : A
\end{aligned}$$

 λP :

$$\begin{aligned}
s &::= * \mid \square \\
M, N, A, B &::= x \mid MN \mid \lambda x : A. M \mid \overline{\Pi x : A. B} \mid s \\
\Gamma &::= \cdot \mid \Gamma, x : A \\
\mathcal{J} &::= \Gamma \vdash M : A
\end{aligned}$$

$$a : *, b : *, f : a \rightarrow b, x : a \vdash fx : b$$

λP

STT versus λP syntax

STT:

$$\begin{aligned} A, B &::= a \mid A \rightarrow B \\ M, N &::= x \mid MN \mid \lambda x : A. M \\ \Gamma &::= \cdot \mid \Gamma, x : A \\ \mathcal{J} &::= \Gamma \vdash M : A \end{aligned}$$

λP :

$$\begin{aligned} s &::= * \mid \square \\ M, N, A, B &::= x \mid MN \mid \lambda x : A. M \mid \overline{A \rightarrow B} \mid \Pi x : A. B \mid s \\ \Gamma &::= \cdot \mid \Gamma, x : A \\ \mathcal{J} &::= \Gamma \vdash M : A \end{aligned}$$

$$\overline{\Gamma} \quad \overline{M} \quad \overline{A} \\ a : *, b : *, f : a \rightarrow b, x : a \vdash fx : b$$

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A. M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A. M \mid \Pi x : A. M \mid * \mid \square$$

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A. M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A. M \mid \Pi x : A. M \mid * \mid \square$$

box does not have a type
→ no rule for box

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A. M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A. M \mid \Pi x : A. M \mid *$$

box does not have a type
→ no rule for box

variables have two rules

STT versus λP

STT

3 rules

$$x \mid MN \mid \lambda x : A. M$$

λP

7 rules

$$x \mid MN \mid \lambda x : A. M \mid \Pi x : A. M \mid *$$

box does not have a type
→ no rule for box

variables have two rules

conversion rule

typing rules

λP

$\frac{}{\vdash * : \square}$

typing rules

λP

$\overline{\vdash * : \square}$

axiom rule

start rule

λP

$$\overline{\vdash * : \square}$$

axiom rule

start rule

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

product rule

typing rules

STT

λP

$$\overline{\vdash * : \square}$$

axiom rule

start rule

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

product rule

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A. M : A \rightarrow B}$$

abstraction rule

typing rules

STT

λP

$$\overline{\vdash * : \Box}$$

axiom rule

start rule

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

product rule

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A. M : A \rightarrow B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B}$$

abstraction rule

typing rules

STT

λP

$$\overline{\vdash * : \square}$$

axiom rule

start rule

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

product rule

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A. M : A \rightarrow B}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B}$$

abstraction rule

typing rules (continued)

STT

λP

$$\frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

application rule

typing rules (continued)

STT

$$\frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

λP

$$\frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

application rule

typing rules (continued)

STT

$$\frac{\Gamma \vdash F : A \rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B}$$

λP

$$\frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

application rule

truesvec : $\Pi n : \text{nat}. \text{vec } n$
truesvec (S (S O)) : $\text{vec } (\text{S } (\text{S } \text{O}))$

typing rules (continued)

STT

λP

$$\frac{}{\Gamma \vdash \textcolor{red}{x} : A} (x : A) \in \Gamma$$

variable rule

typing rules (continued)

STT

(context Γ is a set)

$$\frac{}{\Gamma \vdash \textcolor{red}{x} : A} (x : A) \in \Gamma$$

λP

(context Γ is a list)

correct: $a : *, x : a \vdash x : a$

incorrect: $x : a, a : * \vdash x : a$

variable rule

typing rules (continued)

STT

(context Γ is a set)

$$\frac{}{\Gamma \vdash x : A} (x : A) \in \Gamma$$

λP

(context Γ is a list)

correct: $a : *, x : a \vdash x : a$

incorrect: $x : a, a : * \vdash x : a$

$$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}$$

variable rule

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

weakening rule

the final typing rule

$\text{vecappend} : \prod n_1 : \text{nat}. \prod n_2 : \text{nat}.$

$\text{vec } n_1 \rightarrow \text{vec } n_2 \rightarrow \text{vec } (\text{add } n_1 \ n_2)$

$\text{vecappend } 3 \ 4 \ v_1 \ v_2 : \text{vec } (\text{add } 3 \ 4)$

$\text{vecappend } 3 \ 4 \ v_1 \ v_2 : \text{vec } 7$

the final typing rule

$\text{vecappend} : \Pi n_1 : \text{nat}. \Pi n_2 : \text{nat}.$

$\text{vec } n_1 \rightarrow \text{vec } n_2 \rightarrow \text{vec } (\text{add } n_1 \ n_2)$

$\text{vecappend } 3 \ 4 \ v_1 \ v_2 : \text{vec } (\text{add } 3 \ 4)$

$\text{vecappend } 3 \ 4 \ v_1 \ v_2 : \text{vec } 7$

λP

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash A' : s}{\Gamma \vdash M : A'} A =_{\beta} A'$$

conversion rule

A and A' are convertible

$=_{\beta}$ is defined on preterms

$$M, F, A, B ::= x \mid FM \mid \lambda x : A. M \mid \Pi x : A. B \mid s$$

$$s ::= * \mid \square$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

$$\frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash A' : s}{\Gamma \vdash M : A'} A =_{\beta} A'$$

the lambda cube

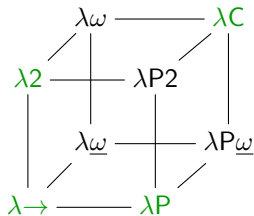
only the product rules differ:

$$\lambda \rightarrow \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x : A. B : *}$$

$$\lambda P \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

$$\lambda 2 \frac{\Gamma \vdash A : s \quad \Gamma, x : A \vdash B : *}{\Gamma \vdash \Pi x : A. B : *}$$

$$\lambda C \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_2}$$



$$\frac{}{\vdash s_1 : s_2} (s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_3} (s_1, s_2, s_3) \in \mathcal{R}$$

pure type systems

$$\frac{}{\vdash_{s_1 : s_2} (s_1, s_2) \in \mathcal{A}}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_3} (s_1, s_2, s_3) \in \mathcal{R}$$

$\mathcal{S}$
sorts

$\mathcal{A}$
axioms

$\mathcal{R}$
rules

$\lambda \rightarrow$

$\{*, \Box\}$

$\{(*, \Box)\}$

$\{(*, *, *)\}$

λP

$\{*, \Box\}$

$\{(*, \Box)\}$

$\{(*, *, *), (*, \Box, \Box)\}$

pure type systems

$$\frac{}{\vdash_{s_1 : s_2} (s_1, s_2) \in \mathcal{A}}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_3} (s_1, s_2, s_3) \in \mathcal{R}$$

$\mathcal{S}$
sorts

$\mathcal{A}$
axioms

$\mathcal{R}$
rules

$\lambda*$	$\{*\}$	$\{(*, *)\}$	$\{(*, *, *)\}$
$\lambda\rightarrow$	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *)\}$
λP	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square)\}$

pure type systems

$$\frac{}{\vdash_{s_1 : s_2} (s_1, s_2) \in \mathcal{A}}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_3} (s_1, s_2, s_3) \in \mathcal{R}$$

	$\mathcal{S}$ sorts	$\mathcal{A}$ axioms	$\mathcal{R}$ rules
$\lambda*$	$\{*\}$	$\{(*, *)\}$	$\{(*, *, *)\}$
$\lambda \rightarrow$	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *)\}$
λP	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square)\}$
λC	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square),$ $(\square, *, *), (\square, \square, \square)\}$

pure type systems

$$\frac{}{\vdash s_1 : s_2} (s_1, s_2) \in \mathcal{A}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A. B : s_3} (s_1, s_2, s_3) \in \mathcal{R}$$

	$\mathcal{S}$ sorts	$\mathcal{A}$ axioms	$\mathcal{R}$ rules
$\lambda*$	$\{*\}$	$\{(*, *)\}$	$\{(*, *, *)\}$
$\lambda \rightarrow$	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *)\}$
λP	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square)\}$
λC	$\{*, \square\}$	$\{(*, \square)\}$	$\{(*, *, *), (*, \square, \square),$ $(\square, *, *), (\square, \square, \square)\}$
$\lambda PRED$	$\{*_s, \square_s, *_p, \square_p\}$	$\{(*_s, \square_s), (*_p, \square_p)\}$	$\{(*_s, *_s, *_s), (*_s, \square_s, \square_s),$ $(*_p, *_p, *_p), (*_s, *_p, *_p)\}$

example PTS type derivation

$$a : *, x : a \vdash (\lambda y : a. y) x : a$$

example PTS type derivation

$$a : *, x : a \vdash (\lambda y : a. y)x : a$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\frac{a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a}{a : *, x : a \vdash (\lambda y : a. y)x : a}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\frac{a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : *}{a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a} \quad a : *, x : a \vdash x : a$$

$$\frac{}{a : *, x : a \vdash (\lambda y : a. y)x : a}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\begin{array}{c}
 \textcolor{red}{a : *}, \textcolor{red}{x : a} \vdash \textcolor{red}{a : *} \\
 \hline
 a : *, x : a, \textcolor{green}{y : a} \vdash \textcolor{green}{y : a} \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

$$\begin{array}{ccc}
 \frac{}{\vdash * : \Box} & \frac{\textcolor{red}{\Gamma \vdash A : s}}{\Gamma, \textcolor{green}{x : A} \vdash \textcolor{green}{x : A}} & \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}
 \end{array}$$

$$\begin{array}{cc}
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} & \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

$$\begin{array}{c}
 \hline
 \vdash * : \square \\
 \hline
 \Gamma \vdash A : s \\
 \hline
 \Gamma, x : A \vdash x : A
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \vdash M : A \quad \Gamma \vdash B : s \\
 \hline
 \Gamma, y : B \vdash M : A
 \end{array}$$

$$\begin{array}{c}
 \Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s \\
 \hline
 \Gamma \vdash \lambda x : A. M : \Pi x : A. B
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A \\
 \hline
 \Gamma \vdash FM : B[x := M]
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 \textcolor{red}{\vdash * : \Box} \\
 \hline
 \textcolor{green}{a} : * \vdash \textcolor{green}{a} : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

$$\begin{array}{ccc}
 \frac{}{\vdash * : \Box} & \frac{\textcolor{red}{\Gamma \vdash A : s}}{\Gamma, \textcolor{green}{x} : A \vdash \textcolor{green}{x} : A} & \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}
 \end{array}$$

$$\begin{array}{cc}
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} & \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 \overline{\vdash * : \square} \\
 \hline
 a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

$$\begin{array}{c}
 \overline{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}
 \end{array}$$

$$\begin{array}{c}
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{}{\vdash * : \square} \quad \frac{}{\vdash * : \square} \\
 \hline
 a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 \frac{a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a}{a : *, x : a \vdash (\lambda y : a. y)x : a}
 \end{array}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{}{\overline{\vdash * : \square}} \\
 \hline
 a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

(next slide)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A} \\
 \hline
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{}{\overline{\vdash * : \square}} \\
 \hline
 a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

(next slide)

$$\frac{}{\overline{\vdash * : \square}} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{}{\overline{\vdash * : \square}} \\
 \hline
 a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}
 \quad
 \begin{array}{c}
 \text{(next slide)} \\
 \vdots \\
 \frac{}{\overline{\vdash * : \square}} \\
 \hline
 a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash x : a
 \end{array}$$

$$\frac{}{\overline{\vdash * : \square}} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}
 \quad
 \begin{array}{c}
 \text{(next slide)} \\
 \vdots \\
 \overline{\vdash * : \square} \\
 \hline
 a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash x : a
 \end{array}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A. B : s}{\Gamma \vdash \lambda x : A. M : \Pi x : A. B} \quad \frac{\Gamma \vdash F : \Pi x : A. B \quad \Gamma \vdash M : A}{\Gamma \vdash FM : B[x := M]}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \text{(next slide)} \\
 \frac{a : *, x : a \vdash a : *}{a : *, x : a, y : a \vdash y : a} \quad \frac{\vdots}{a : *, x : a \vdash a \rightarrow a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \frac{a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : *}{a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a} \quad \frac{a : * \vdash a : *}{a : *, x : a \vdash x : a} \\
 \frac{a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a}{a : *, x : a \vdash (\lambda y : a. y)x : a}
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad a : *, x : a \vdash a \rightarrow a : * \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

(next slide)

$$\begin{array}{c}
 \frac{\overline{x : a, y : a \vdash y : a}}{x : a \vdash (\lambda y : a. y) : a \rightarrow a} \quad \frac{}{x : a \vdash x : a} \\
 \hline
 x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

example PTS type derivation

$$\begin{array}{c}
 \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash a : * \\
 \hline
 a : *, x : a, y : a \vdash y : a \quad \textcolor{red}{a : *, x : a \vdash a \rightarrow a : *} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y) : a \rightarrow a \quad a : *, x : a \vdash x : a \\
 \hline
 a : *, x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

(next slide)

$$\begin{array}{c}
 \frac{\overline{x : a, y : a \vdash y : a}}{x : a \vdash (\lambda y : a. y) : a \rightarrow a} \quad \frac{\overline{x : a \vdash x : a}}{x : a \vdash (\lambda y : a. y)x : a} \\
 \hline
 x : a \vdash (\lambda y : a. y)x : a
 \end{array}$$

example PTS type derivation (continued)

$$a : *, x : a \vdash a \rightarrow a : *$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$
$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

example PTS type derivation (continued)

$$\frac{a : *, x : a \vdash a \rightarrow a : *}{\Pi y : a. a}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

example PTS type derivation (continued)

$$\frac{a : * \vdash (\Pi y : a. a) : * \quad a : * \vdash a : *}{a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{a : * \vdash a : * \quad a : *, y : a \vdash a : *}{a : * \vdash (\Pi y : a. a) : *} \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\frac{}{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \textcolor{red}{\vdash * : \Box} \\
 \hline
 \textcolor{green}{a} : * \vdash \textcolor{green}{a} : * \quad a : *, y : a \vdash a : * \\
 \hline
 a : * \vdash (\Pi y : a. a) : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\begin{array}{c}
 \hline
 \vdash * : \Box
 \end{array}
 \quad
 \begin{array}{c}
 \textcolor{red}{\Gamma \vdash A : s} \\
 \hline
 \Gamma, \textcolor{green}{x} : A \vdash \textcolor{green}{x} : A
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \vdash M : A \quad \Gamma \vdash B : s \\
 \hline
 \Gamma, y : B \vdash M : A
 \end{array}$$

$$\begin{array}{c}
 \Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s \\
 \hline
 \Gamma \vdash \Pi x : A. B : s
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \overline{\vdash * : \square} \\
 \hline
 a : * \vdash a : * \quad a : *, y : a \vdash a : * \\
 \hline
 a : * \vdash (\Pi y : a. a) : * \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\begin{array}{c}
 \overline{\vdash * : \square} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A} \\
 \\
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \overline{\vdash * : \Box} \quad \quad \quad a : * \vdash a : * \quad a : * \vdash a : * \\
 \hline
 a : * \vdash a : * \quad \quad \quad a : *, y : a \vdash a : * \\
 \hline
 a : * \vdash (\Pi y : a. a) : * \quad \quad \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\begin{array}{c}
 \overline{\vdash * : \Box} \quad \quad \quad \Gamma \vdash A : s \quad \quad \quad \Gamma \vdash M : A \quad \Gamma \vdash B : s \\
 \hline
 \Gamma, x : A \vdash x : A \quad \quad \quad \Gamma, y : B \vdash M : A \\
 \hline
 \Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s \\
 \hline
 \Gamma \vdash \Pi x : A. B : s
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\overline{\vdash * : \square} \quad \overline{a : * \vdash a : *}}{a : * \vdash a : *} \quad \frac{\overline{a : * \vdash a : *}}{a : *, y : a \vdash a : *} \\
 \hline
 \frac{a : * \vdash (\Pi y : a. a) : * \quad a : * \vdash a : *}{a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *}
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\overline{\Gamma \vdash A : s} \quad \overline{\Gamma, x : A \vdash x : A}}{\Gamma, x : A \vdash x : A} \quad \frac{\overline{\Gamma \vdash M : A} \quad \overline{\Gamma \vdash B : s}}{\Gamma, y : B \vdash M : A} \\
 \hline
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad a : * \vdash a : * \\
 \hline
 \frac{a : * \vdash a : * \quad a : *, y : a \vdash a : *}{a : * \vdash (\Pi y : a. a) : *} \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A} \\
 \\
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\overline{\vdash * : \square}}{a : * \vdash a : *} \quad \frac{\textcolor{red}{\vdash * : \square}}{a : * \vdash \textcolor{green}{a} : *} \\
 \hline
 \frac{a : * \vdash a : * \quad a : *, y : a \vdash a : *}{a : * \vdash (\Pi y : a. a) : *} \quad a : * \vdash a : * \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \square}} \quad \frac{\textcolor{red}{\Gamma \vdash A : s}}{\Gamma, \textcolor{green}{x} : A \vdash \textcolor{green}{x} : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A} \\
 \\
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{}{\overline{\vdash * : \Box}} \quad \frac{}{\overline{\vdash * : \Box}} \\
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \quad \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \\
 \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \quad \frac{a : *, y : a \vdash a : *}{\overline{a : *, y : a \vdash a : *}} \\
 \frac{a : * \vdash (\Pi y : a. a) : *}{\overline{a : * \vdash (\Pi y : a. a) : *}} \quad a : * \vdash a : * \\
 \frac{a : *, x : a \vdash \overbrace{a \rightarrow a}^{\Pi y : a. a} : *}{\overline{a : *, x : a \vdash \Pi y : a. a : *}}
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{\Gamma \vdash A : s}{\overline{\Gamma, x : A \vdash x : A}} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\overline{\Gamma, y : B \vdash M : A}} \\
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\overline{\Gamma \vdash \Pi x : A. B : s}}
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{}{\overline{\vdash * : \Box}} \quad \frac{}{\overline{\vdash * : \Box}} \\
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \quad \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \\
 \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \quad \frac{a : *, y : a \vdash a : *}{\overline{a : *, y : a \vdash a : *}} \quad \frac{\vdash * : \Box}{\overline{\vdash * : \Box}} \\
 \frac{a : * \vdash (\Pi y : a. a) : *}{\overline{a : * \vdash (\Pi y : a. a) : *}} \quad \frac{a : * \vdash a : *}{\overline{a : * \vdash a : *}} \\
 \frac{a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *}{\overline{a : *, x : a \vdash \Pi y : a. a : *}}
 \end{array}$$

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{\Gamma \vdash A : s}{\overline{\Gamma, x : A \vdash x : A}} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\overline{\Gamma, y : B \vdash M : A}} \\
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\overline{\Gamma \vdash \Pi x : A. B : s}}
 \end{array}$$

example PTS type derivation (continued)

$$\begin{array}{c}
 \frac{}{\overline{\vdash * : \Box}} \quad \frac{\overline{\vdash * : \Box}}{a : * \vdash a : *} \quad \frac{\overline{\vdash * : \Box}}{a : * \vdash a : *} \\
 \hline
 \frac{a : * \vdash a : * \quad a : *, y : a \vdash a : *}{a : * \vdash (\Pi y : a. a) : *} \quad \frac{\overline{\vdash * : \Box}}{a : * \vdash a : *} \\
 \hline
 a : *, x : a \vdash \underbrace{a \rightarrow a}_{\Pi y : a. a} : *
 \end{array}$$

$$\frac{}{\overline{\vdash * : \Box}} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, y : B \vdash M : A}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A. B : s}$$

conclusion

computing the sort of a type

$$\text{type}_{\Gamma}(\lambda x : A. M) = \Pi x : A. \text{type}_{\Gamma, x:A}(M)$$

$$\text{type}_{\Gamma}(\Pi x : A. B) = \text{type}_{\Gamma, x:A}(B)$$

$$\text{type}_{\Gamma}(A \rightarrow B) = \text{type}_{\Gamma}(B)$$

$$\text{type}_{\Gamma}(*) = \square$$

$$\text{type}_{\Gamma}(x) = \Gamma(x)$$

$$\text{type}_{\Gamma}(FM) = \dots$$

summary

- ▶ predicate logic
- ▶ dependent types
 - ▶ vectors
 - ▶ Curry-Howard
- ▶ λP
 - ▶ lambda cube
 - ▶ PTSs
- ▶ detours

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