A logic for parametric polymorphism
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Parametricity: Stracheys vs Reynolds

- Strachey: Uniform under simple types
  - not a formal definition
  - this paper: same type + same erasure = strachey parametric
- Reynolds: Relational parametricity
  - This notion of binary relational parametricity in the reynolds sense can be extended. Either to "kripke" semantics or N-ary relational parametricity.
  - will be seen as provable equality in this logic
- Conjecture: Strachey implies Reynolds, but not Reynolds implies Strachey. (Or: Dinaturality is a consequence of the schema expressing binary relational parametricity)

Goal of the paper

- Provide a formal system for arguments that exploit relational parametricity. (Section 2)
  - Standard rules for the connectives and quantifiers
  - Axioms and rules for the equational part of System-F
  - A schema to express relational parametricity
- Provide a system to reason with denotational semantics (recursion and arithmatic, future work)

Recap: λ-Calculus vs System-F

Types:	$A, B := \ a \mid A \to B$
Terms:	$M, N := \ x \mid \lambda x : A.M \mid MN$
Context:	$\Gamma := \ \cdot \mid \Gamma, x : A$

$$\textsc{Var}\quad \frac{}{\Gamma \vdash x : A} \quad \texttt{for}\ (x : A) \in \Gamma$$

$$\textsc{Abs}\quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.M) : A \to B}$$

$$\textsc{App}\quad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

Recap: λ-Calculus vs System-F

Types:	$A, B := \ a \mid A \to B \mid \forall a.A$
Terms:	$M, N := \ x \mid \lambda x : A.M \mid MN \mid \Lambda a.M \mid MA$
Context:	$\Gamma := \ \cdot \mid \Gamma, x : A$

Var

$$\frac{}{\Gamma \vdash x : A} \quad \texttt{for } (x : A) \in \Gamma$$

Abs

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash (\lambda x : A.M) : A \to B}$$

App

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

Type Abs

$$\frac{\Gamma, x : A \vdash M : A}{\Gamma \vdash (\Lambda x : a.M) : A}$$

Type App

$$\frac{\Gamma \vdash M : \forall a.A}{\Gamma \vdash MB : A[a := B]}$$

System-F: differences in notation

$$a := X$$

$$M, N := u, t$$

System-F: à la Plotkin & Abadi

`Types:`	$A, B :=$	$X \mid A \to B \mid \forall X.A$
`Terms:`	$u, t :=$	$x \mid \lambda x : A.t \mid u(t) \mid \Lambda X.t \mid t(A)$
`Context:`	$\Gamma :=$	$\cdot \mid \Gamma, x : A$

$$\textsc{Var}\quad \frac{}{\Gamma \vdash x : A} \quad \texttt{for}\ (x : A) \in \Gamma$$

$$\textsc{Abs}\quad \frac{\Gamma, x : A \vdash u : B}{\Gamma \vdash (\lambda x : A.u) : A \to B} \qquad \textsc{App}\quad \frac{\Gamma \vdash u : A \to B \quad \Gamma \vdash t : A}{\Gamma \vdash u(t) : B}$$

$$\textsc{Type Abs}\quad \frac{\Gamma, x : A \vdash u : A}{\Gamma \vdash (\Lambda x : X.u) : A} \qquad \textsc{Type App}\quad \frac{\Gamma \vdash u : \forall X.A}{\Gamma \vdash u(B) : A[X := B]}$$

Syntax: Formulae

$$
\begin{aligned}
\phi, \psi ::= \; & R(t, u) && \text{(A typing relation between terms t and u)} \\
& (t =_A u) && \text{(Equality relation on type A between t and u)}
\end{aligned}
$$

Syntax: Formulae

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\begin{aligned}
\phi, \psi ::= \;& R(t, u) && \text{(A typing relation between terms t and u)} \\
& (t =_A u) && \text{(Equality relation on type A between t and u)} \\
& \forall x : A.\phi && (\forall \text{ values}) \\
& \forall X.\phi && (\forall \text{ types}) \\
& \forall R \subset A \times B.\phi && (\forall \text{ relations between A and B}) \\
& \exists x : A.\phi && (\exists \text{ value}) \\
& \exists X.\phi && (\exists \text{ type}) \\
& \exists R \subset A \times B.\phi && (\exists \text{ relation between A and B})
\end{aligned}
$$

Syntax: Formulae

$$
\begin{array}{rll}
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& \exists X.\phi & (\exists \text{ type}) \\
& \exists R \subset A \times B.\phi & (\exists \text{ relation between A and B}) \\
& \phi \supset \psi & (\phi \text{ implies } \psi) \\
& \phi \wedge \psi & \text{(conjunction)} \\
& \phi \vee \psi & \text{(disjunction)} \\
& \bot & \text{(falsum)}
\end{array}
$$

Syntax: Definable Relations

Definable relation

$$\rho ::= (x : A, y : B).\phi[x, y]$$

$$\rho \subset A \times B$$

ρ is a definable relation between A and B

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Examples:

$$eq_A = (x : A, y : A).(x =_A y)$$

$$\langle f \rangle = (x : A, y : B).(y =_B f(x))$$

if f is of type $A \to B$ then f is a well formed relation

Syntax: Usage ρ

$$\rho(u, t)$$

or

$$u \rho t$$

Semantics

Meaning of typing these relations

When the formula type they are well-formed and we can extract which terms have which types.

E

finite sequence of X or x:A where all x are unique

G : REnv

finite sequence of definable relations between types

Semantics: Typing Judgements

Well formed term

$$\frac{}{E \vdash x : A}$$

Well formed type

$$\frac{}{E \vdash A : Type}$$

Well formed relation environment

$$\frac{}{E \vdash G : REnv}$$

where $\forall R \subset A \times B \in G$, A and B must be well formed given E.

Semantics: Relations

Relation from type A to A

$$\frac{E \vdash t : A \quad E \vdash u : A \quad E \vdash G : REnv}{E; G \vdash t =_A u : Prop}$$

Relations from type A to B

$$\frac{E \vdash t : A \quad E \vdash u : B \quad E \vdash G : REnv \text{ where } R \subset A \times B \in G}{E; G \vdash R(t, u) : Prop}$$

Definable relations

$$\frac{E; G, x : A, y : B \vdash \phi : Prop}{E; G \vdash (x : A, y : B.\phi \subset A \times B) : Prop}$$

Semantics: Propositional Formulae

$\forall X.\phi$

$$\frac{E, X; G \vdash \phi : Prop}{E; G \vdash (\forall X.\phi) : Prop}$$

$\forall R \subset A \times B.\phi$

$$\frac{E; G, R \subset A \times B \vdash \phi : Prop}{E; G \vdash (\forall R \subset A \times B.\phi) : Prop}$$

$\phi \wedge \psi$

$$\frac{E; G \vdash \phi : Prop \quad E; G \vdash \psi : Prop}{E; G \vdash (\phi \wedge \psi) : Prop}$$

etc.

Semantics: Substitution

Combining definable relations

- If $\rho \subset A \times B$ and $\rho' \subset A' \times B'$
- Then $(\rho \to \rho') \subset (A \to A') \times (B \to B')$ is defined by $f(\rho \to \rho')g \equiv \forall x : A \forall x' : A'.(x \rho x' \supset f(x)\rho' g(x'))$
- Thus if ρ and ρ' are definable relations $\rho \to \rho'$ is definable as well

- If $\rho \subset A \times B$
- Then $\forall Y \forall Z \forall R \subset Y \times Z.\rho \subset (\forall Y.A)x(\forall Z.B)$ is defined by $y(\forall Y \forall Z \forall R \subset Y \times Z.\rho)z \equiv \forall Y \forall Z \forall R \subset Y \times Z.((yY)\rho(zZ))$
- Thus, if ρ is well-formed given $E, Y, Z; G, (R \subset Y \times Z)$ then $\forall Y \forall Z \forall R \subset Y \times Z.p$ is well-formed given $E; G$.

Semantics: Substitution

Vectors

Let $\overrightarrow{X} = X_1 ... X_n$ and $\overrightarrow{\rho} = \rho_1 ... \rho_n$

Definition substitution

By cases:

- if A is X_i then $A[\overrightarrow{\rho}]$ is ρ_i
- if A is $A' \to A''$ then $A[\rho] = A'[\rho] \to A''[\rho]$
- if A is $\forall Y. A'[\overrightarrow{X}, Y]$ then $A'[\overrightarrow{\rho}] = \forall Y \forall Z \forall R \subset Y \times Z. A'[\overrightarrow{\rho}, R]$

If each ρ_i is well formed given $E; G$, then so is $A[\overrightarrow{\rho}]$

Semantics: Natural Deduction

Intuistionistic Logic

$$\frac{\Gamma, \phi \vdash_{E;G} \psi}{\Gamma \vdash_{E;G} \phi \supset \psi} \qquad \frac{\Gamma \vdash_{E;G} \phi \supset \psi \quad \Gamma \vdash_{E;G} \phi}{\Gamma \vdash_{E;G} \psi} \qquad \frac{\Gamma \vdash_{E,X;G} \phi[X]}{\Gamma \vdash_{E;G} \forall X.\phi[X]}$$

$$\frac{\Gamma \vdash_{E;G} \forall X.\phi[X] \quad E \vdash A : \mathit{Type}}{\Gamma \vdash_{E;G} \phi[A]} \qquad \frac{\Gamma \vdash_{E;G,R\subset A\times B} \phi[R]}{\Gamma \vdash_{E;G} \forall R \subset A \times B.\phi[R]}$$

$$\frac{\Gamma \vdash_{E;G} \forall R \subset A \times B.\phi[R] \quad E;G \vdash \rho \subset A \times B}{\Gamma \vdash_{E;G} \phi[\rho]}$$

Semantics: Axioms

Reflexivity

$$\forall X \forall x : X.(x =_X x)$$

Substitution

$$\forall X \forall Y \forall R \subset X \times Y \forall x : X \forall x' : X \forall y : Y \forall y' : Y.$$
$$(R(x, y) \wedge x =_X x' \wedge y =_Y y') \supset R(x', y')$$

Semantics: Schemas

Congruence

Terms:

$$(\forall x : A.t =_B u) \supset (\lambda x : A.t) =_{A \to B} (\lambda x : A.u)$$

Types:

$$(\forall X.t =_B u) \supset (\Lambda X.t) =_{\forall X.B} (\Lambda X.u)$$

Semantics: Equalities

β-equalities

Terms:

$$\forall x : A.((\lambda x : A.t)x =_B t)$$

Types:

$$\forall x.((\Lambda X.t)x =_B t$$

η-equalities

Terms:

$$\forall A \forall B \forall f : A \to B.((\lambda x : A.fx) =_{A \to B} f)$$

Types:

$$\forall f : (\forall X.B).((\Lambda X.fX) =_{\forall X.B} f)$$

Semantics: Parametricity Schema

Full definition

$$\forall Y_1 ... Y_n \forall u : (\forall X . A[X, Y_1, ..., Y_n]).$$
$$u(\forall X . A[X, eq_1, ..., eq_n])u$$

Simplifies to

$$\forall (u : X . A[X]) . \forall Y \forall Z \forall R \subset Y \times Z .$$
$$u(Y) A[R] u(Z)$$

Said plainly

The relation R is kept between parametric instantiation of u with Y and Z

Identity Extension Lemma

Identity Extension Lemma (Lemma 1)

For any $A[\overrightarrow{X}]$ with free variables in $\overrightarrow{X}$ it is provable in the logic that:

$$\forall \overrightarrow{X} \forall u : A.(uA[eq_{\overrightarrow{X}}]v \equiv (u =_A v))$$

Logical Relations Lemma

Logical Relations Lemma (Lemma 2)

Let $A_1[\overrightarrow{X}], ..., A_m[\overrightarrow{X}], B[\overrightarrow{X}]$ have free type variables in $\overrightarrow{X}$. Suppose $t[x_1, ..., x_m] : B$ given the environment $\overrightarrow{X}, x_1 : A_1, ..., x_m : A_m$. Then the following is provable in the logic without using the parametricity schema:

$$\forall \overrightarrow{X} \forall \overrightarrow{Y} \forall \overrightarrow{R} \subset \overrightarrow{X} \times \overrightarrow{Y}$$
$$\forall x_1 : A_1[\overrightarrow{X}], \ldots, x_m : A_m[\overrightarrow{X}]$$
$$\forall y_1 : A_1[\overrightarrow{Y}], \ldots, y_m : A_m[\overrightarrow{Y}].$$
$$(x_1 A_1[\overrightarrow{R}] y_1 \wedge \cdots \wedge x_m A_m[\overrightarrow{R}] y_m) \supset t[x_1, ..., x_m] A[\overrightarrow{R}] t[y_1, ..., y_m]$$

Dinaturality

Dinaturality

- bi(endo)functors $A, B : C^{op} \times C \to C$
- can be seen as set of morphisms on functors to itself

$$\begin{tikzcd}
 & A[X,X] \arrow[r, "t_X"] & B[X,X] \arrow[dr, "B[\mathrm{id}_X, f]"] & \\
A[Y,X] \arrow[ur, "A[f, \mathrm{id}_X]"] \arrow[dr, "A[\mathrm{id}_Y, f]"'] & & & B[X,Y] \\
 & A[Y,Y] \arrow[r, "t_Y"'] & B[Y,Y] \arrow[ur, "B[f, \mathrm{id}_X]"'] &
\end{tikzcd}$$

Dinaturality simplified

$$f \circ \mathbf{u}_X = \mathbf{u}_Y \circ f$$

Graph Lemma

Graph Lemma (Lemma 3)

Let $A[\overrightarrow{X}, \overrightarrow{Y}]$ be a type all of whose free variables appear in the list $\overrightarrow{X}, \overrightarrow{Y}$ and such that all type variables in $\overrightarrow{X}$ have only negative occurrences, and all type variables in $\overrightarrow{Y}$ have only positive occurrences. Then it can be proved in the logic that:

$$\forall \overrightarrow{X}, \overrightarrow{Y}, \overrightarrow{X'}, \overrightarrow{Y'}. \forall \overrightarrow{f} : X' \to X. \forall \overrightarrow{g} : Y \to Y'. (\langle A[\overrightarrow{f}, \overrightarrow{g}] \rangle = A[\langle \overrightarrow{f^{op}} \rangle, \langle \overrightarrow{g} \rangle])$$

- Graph of the functor commutes with the functor of the graph.
- Note: 'op' stands for the opposite, on $\rho \subset A \times B$, the opposite is defined as $\rho^{op} = (y : B, x : A).x \rho y$

Easier graph lemma

simplified Graph Lemma

For any functor $A[X]$, with X occurring positively we have the following statement:

$$\forall \overrightarrow{X}, \overrightarrow{X'}. \forall \overrightarrow{f} : X \to X'.(\langle A[\overrightarrow{f}] \rangle = A[\langle \overrightarrow{f} \rangle])$$

- graph of the functor = functor of the graph

ALTERNATIVE: Per model

We could have started from the Per (or Per_D) model.

- normal per models use universal types as the intersection of all the instances
- so polymorphism could be modeled as having the same realisor as any other type

Looking ahead

In section 3, the paper starts generating properties of this logic using the lemmas shown above, some of the theorems (properties) are shown below:

- unit type is inhabited
- taking the graph of the fst, snd of a pair is the same as the graph of the pair
- this logic is distributive over the relations
- binary sums are expressible, using the identity expansion lemma
- . . . (non exhaustive list)

Looking back

What did we actually do during this talk?

- Introduced a logic system for parametric polymorphism in System-F
- Looked into category theory (Dinaturality, co- and contra- variant)
- Introduced some lemmas to ~~prove~~ (read: reason about) properties
- Had fun!

Q&A

Are there any questions?