Realizability and Parametricity in Pure Type Systems

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Overview

- Recap previous sections
  - Second-level terms
  - Lifting
  - Projection
- Parametricity
- Realizability
- Realizability to Parametricity
- Third-level terms
- Parametricity to Realizability

Recap: Second-level terms

Definition (Second-level system)

Given a PTS $P = (\mathcal{S}, \mathcal{A}, \mathcal{R})$, define (P^2). That is, $P^2 = (\mathcal{S}^2, \mathcal{A}^2, \mathcal{R}^2)$, where

$$
\begin{aligned}
\mathcal{S}^2 &= \mathcal{S} \cup \{\lceil s \rceil \mid s \in S\} \\
\mathcal{A}^2 &= \mathcal{A} \cup \{(\lceil s_1 \rceil, \lceil s_2 \rceil) \mid (s_1, s_2) \in \mathcal{A}\} \\
\mathcal{R}^2 &= \mathcal{R} \cup \{(\lceil s_1 \rceil, \lceil s_2 \rceil, \lceil s_3 \rceil), (s_1, \lceil s_3 \rceil, \lceil s_3 \rceil) \ \mid \ (s_1, s_2, s_3) \in \mathcal{R}\} \cup \\
&\qquad \{(s_1, \lceil s_2 \rceil, \lceil s_2 \rceil) \mid (s_1, s_2) \in \mathcal{A}\}
\end{aligned}
$$

Example

The PTS F^2 is an example for P^2, and will be used in further examples.

Recap: Intuition of 2nd level

- *Type*: term inhabiting a first-level sort ($\Gamma : A : s$)
- *Program*: inhabitant of a *type* ($\Gamma : A : B : s$)
- *Formula*: inhabitant of a lifted sort ($\Gamma : A : \lceil s \rceil$)
- *Proof*: inhabitant of a *formula* ($\Gamma : A : B : \lceil s \rceil$)

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So, *Type* and *Program* are first-level terms, *Formula* and *Proof* are second-level terms.

Recap: Lifting

Definition (Lifting)

$$
\begin{array}{ll}
\lceil x \rceil & = \mathring{x} \\
\lceil s \rceil & = \lceil s \rceil \\
\lceil \Pi x : A.B \rceil & = \Pi \mathring{x} : \lceil A \rceil . \lceil B \rceil \\
\lceil \lambda x : A.B \rceil & = \lambda \mathring{x} : \lceil A \rceil . \lceil B \rceil \\
\lceil AB \rceil & = \lceil A \rceil \lceil B \rceil \\
\hline
\lceil <> \rceil & = <> \\
\lceil \Gamma, x : A \rceil & = \lceil \Gamma \rceil, \mathring{x} : \lceil A \rceil
\end{array}
$$

Recap: Projection

Definition (Projection)

$$
\begin{array}{ll}
\lfloor x^{\lceil s \rceil} \rfloor & = \dot{x}^s \\
\lfloor \lceil s \rceil \rfloor & = s \\
\lfloor \Pi x^s : A.B \rfloor & = \lfloor B \rfloor \\
\lfloor \Pi x^{\lceil s \rceil} : A.B \rfloor & = \Pi \dot{x}^s : \lfloor A \rfloor . \lfloor B \rfloor \\
\lfloor \lambda x^s : A.B \rfloor & = \lfloor B \rfloor \\
\lfloor \lambda x^{\lceil s \rceil} : A.B \rfloor & = \lambda \dot{x}^s : \lfloor A \rfloor . \lfloor B \rfloor \\
\lfloor (AB)_s \rfloor = & = \lfloor A \rfloor \\
\lfloor (AB)_{\lceil s \rceil} \rfloor & = \lfloor A \rfloor \lfloor B \rfloor \\
\hline
\lfloor <> \rfloor & = <> \\
\lfloor \Gamma, x^s : A \rfloor & = \lfloor \Gamma \rfloor \\
\lfloor \Gamma, x^{\lceil s \rceil} : A \rfloor & = \lfloor \Gamma \rfloor, \dot{x}^s : \lfloor A \rfloor
\end{array}
$$

pause Note : $\lfloor \lceil A \rceil \rfloor = A$

Parametricity

- $\overline{C} \in [\![T]\!]_n$
  An n-tuple of programs $\overline{C}$ satisfies the relation generated by a type T
- $(\!|C|\!)_n$
  A proof that an n-tuple $\overline{C}$ satisfies the relation generated by the type T, for a program C of type T
- The tuple $\overline{A}$ denotes n terms A_i, where A_i is the term A where each free variable x is replaced by a fresh variable x_i.

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Example

If $A = \lambda(x : \alpha).x$ and we are considering a binary relation, then $\overline{A} = (A_1, A_2) = (\lambda(x : \alpha_1).x, \lambda(x : \alpha_2).x)$.

Defining parametricity

In the case where $n = 2$, $\overline{C} = (C_1, C_2)$ and we arrive at the following definition.

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Definition (Parametricity, case n=2)

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\begin{array}{ll}
(C_1, C_2) \in [\![s]\!] & = C_1 \to C_2 \to \lceil s \rceil \\
(C_1, C_2) \in [\![\Pi x{:}A.\ B]\!] & = \Pi(x_1{:}A_1)(x_2{:}A_2).\ \Pi \mathring{x} : (x_1, x_2) \in [\![A]\!]. \\
 & \qquad (C_1 x_1, C_2 x_2) \in [\![B]\!] \\
(C_1, C_2) \in [\![T]\!] & = (\!| T |\!)\, C_1\, C_2,\ \textit{otherwise}
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(C_1, C_2) \in [\![T]\!] & = (\!|T|\!)\, C_1\, C_2,\ \textit{otherwise} \\
\hline
(\!|x|\!) & = \mathring{x} \\
(\!|\lambda x{:}A.\ B|\!) & = \lambda(x_1{:}A_1)(x_2{:}A_2).\ \lambda\mathring{x} : (x_1, x_2) \in [\![A]\!].\ (\!|B|\!) \\
(\!|A\ B|\!) & = (\!|A|\!)\, B_1\, B_2\, (\!|B|\!) \\
(\!|T|\!) & = \lambda(z_1{:}T_1)(z_2{:}T_2).\ (z_1, z_2) \in [\![T]\!],\ \text{otherwise}
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(\!|A\ B|\!) & = (\!|A|\!)\, B_1\, B_2\, (\!|B|\!) \\
(\!|T|\!) & = \lambda(z_1{:}T_1)(z_2{:}T_2).\ (z_1, z_2) \in [\![T]\!],\ \text{otherwise} \\
\hline
(\!|<>|\!) & = <> \\
(\!|\Gamma, x{:}A|\!) & = (\!|\Gamma|\!), x_1{:}A_1, x_2{:}A_2, \mathring{x}{:}(x_1, x_2) \in [\![A]\!]
\end{array}
$$

Parametricity examples

Example

$$
\begin{aligned}
&\overline{x} \in [\![\alpha]\!]_2 \\
&\equiv (x_1, x_2) \in [\![\alpha]\!] \\
&\equiv (\!|\alpha|\!) x_1 x_2 \\
&\equiv \mathring{\alpha} x_1 x_2
\end{aligned}
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$$
\begin{aligned}
&(\!|\alpha : *, x : \alpha|\!)_2 \\
&\equiv (\!|\alpha : *|\!)_2, x_1{:}\alpha_1, x_2{:}\alpha_2, \mathring{x} : (x_1, x_2) \in [\![\alpha]\!]_2 \\
&\equiv \alpha_1{:}*, \alpha_2{:}*, \mathring{\alpha} : (\alpha_1, \alpha_2) \in [\![*]\!], x_1{:}\alpha_1, x_2{:}\alpha_2, \mathring{x} : (x_1, x_2) \in [\![\alpha]\!] \\
&\equiv \alpha_1{:}*, \alpha_2{:}*, \mathring{\alpha} : \alpha_1 \to \alpha_2 \to \lceil * \rceil, x_1{:}\alpha_1, x_2{:}\alpha_2, \mathring{x} : \mathring{\alpha} x_1 x_2
\end{aligned}
$$

Parametricity examples

Example

$\overline{x} \in [\![\alpha]\!]_2$
$\equiv (x_1, x_2) \in [\![\alpha]\!]$
$\equiv (\!|\alpha|\!) x_1 x_2$
$\equiv \mathring{\alpha} x_1 x_2$

$(\!|\alpha : *, x : \alpha|\!)_2$
$\equiv (\!|\alpha : *|\!)_2, x_1{:}\alpha_1, x_2{:}\alpha_2, \mathring{x} : (x_1, x_2) \in [\![\alpha]\!]_2$
$\equiv \alpha_1{:}*, \alpha_2{:}*, \mathring{\alpha} : (\alpha_1, \alpha_2) \in [\![*]\!], x_1{:}\alpha_1, x_2{:}\alpha_2, \mathring{x} : (x_1, x_2) \in [\![\alpha]\!]$
$\equiv \alpha_1{:}*, \alpha_2{:}*, \mathring{\alpha} : \alpha_1 \to \alpha_2 \to \lceil * \rceil, x_1{:}\alpha_1, x_2{:}\alpha_2, \mathring{x} : \mathring{\alpha} x_1 x_2$

Example

Take $T = \alpha \to \alpha = \Pi x : \alpha.\alpha$. Then
$C \in [\![T]\!]_1$
$\equiv \Pi x : \alpha.\Pi \mathring{x} : x \in [\![\alpha]\!].Cx \in [\![\alpha]\!]$
$\equiv \Pi(x : \alpha)(\mathring{x} : \mathring{\alpha} x).\mathring{\alpha} Cx$
$\equiv \Pi(x : \alpha).\mathring{\alpha} x \to \mathring{\alpha}(Cx)$

Abstraction

Theorem (Abstraction)

If $\Gamma \vdash A : B : s$, *then* $(\!|\Gamma|\!) \vdash (\!|A|\!) : (\overline{A} \in [\![B]\!]) : \lceil s \rceil$.

Abstraction

Theorem (Abstraction)

If $\Gamma \vdash A : B : s$, *then* $(\!|\Gamma|\!) \vdash (\!|A|\!) : (\overline{A} \in [\![B]\!]) : \lceil s \rceil$.

Example

Consider $\alpha : *, x : \alpha \vdash x : \alpha : *$.
Then $(\!|\alpha : *, x : \alpha|\!) \vdash (\!|x|\!) : (x_1, x_2) \in [\![\alpha]\!] : \lceil * \rceil$.

Realizability

- A program f realizes $A \in P^2$.
- Notation: $f \Vdash A$ and $\langle p \rangle$.
  - First: $f \Vdash A$: " f realizes A"
  - Then: translate *proof* p to *proof* $\langle p \rangle$; denotes that $\lfloor p \rfloor$ satisfies realizability predicate.
- Difference in A being a first-level or second-level term.
  - $\langle \lambda x^s : A.B \rangle = \lambda x^s : A.\langle B \rangle$
  - $\langle \lambda x^{\lceil s \rceil} : A.B \rangle = \lambda(\mathring{x}^s : \lfloor A \rfloor)(x^{\lceil s \rceil} : \mathring{x} \Vdash A).\langle B \rangle$

Realizability definition

Definition (Realizability)

$$
\begin{aligned}
C \Vdash \lceil s \rceil \quad &= C \to \lceil s \rceil \\
C \Vdash \Pi x^{s} : A.B \quad &= \Pi x^{s} : A.C \Vdash B \\
C \Vdash \Pi x^{\lceil s \rceil} : A.B \quad &= \Pi(\dot{x}^{s} : \lfloor A \rfloor)(x^{\lceil s \rceil} : \dot{x} \Vdash A).C\dot{x} \Vdash B \\
C \Vdash F \quad &= \langle F \rangle C, \ \textit{otherwise}
\end{aligned}
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\hline
\langle x^{\lceil s \rceil} \rangle & = x^{\lceil s \rceil} \\
\langle \lambda x^s : A.B \rangle & = \lambda x^s : A.\langle B \rangle \\
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\langle (AB)_s \rangle & = (\langle A \rangle B)_s \\
\langle (AB)_{\lceil s \rceil} \rangle & = ((\langle A \rangle \lfloor B \rfloor)_s \langle B \rangle)_{\lceil s \rceil} \\
\langle T \rangle & \lambda z^s : \lfloor T \rfloor.z \Vdash T, \textit{otherwise}
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\langle T \rangle & \lambda z^s : \lfloor T \rfloor . z \Vdash T, \textit{otherwise} \\
\hline
\langle \Gamma, x^s : A \rangle & = \langle \Gamma \rangle, x^s : A \\
\langle \Gamma, x^{\lceil s \rceil} : A \rangle & = \langle \Gamma \rangle, \dot{x}^s : \lfloor A \rfloor, x^{\lceil s \rceil} : \dot{x} \Vdash A
\end{array}
$$

Realizability example

Example

$\langle \alpha^{\Box} : *, x^{*} : \alpha \rangle$
$\equiv \langle \alpha^{\Box} : * \rangle, x^{*} : \alpha$
$\equiv \alpha^{\Box} : *, x^{*} : \alpha$

Realizability example

Example

$\langle \alpha^{\square} : *, x^{*} : \alpha \rangle$
$\equiv \langle \alpha^{\square} : * \rangle, x^{*} : \alpha$
$\equiv \alpha^{\square} : *, x^{*} : \alpha$

Example

$\langle \beta^{\lceil \square \rceil} : \lceil * \rceil, y^{\lceil * \rceil} : \beta \rangle$
$\equiv \langle \beta^{\lceil \square \rceil} : \lceil * \rceil \rangle, \dot{y}^{*} : \lfloor \beta \rfloor, y^{\lceil * \rceil} : \dot{y} \Vdash \beta$
$\equiv \dot{\beta}^{\square} : \lfloor \lceil * \rceil \rfloor, \beta^{\lceil \square \rceil} : \dot{\beta} \Vdash \lceil * \rceil, \dot{y}^{*} : \lfloor \beta \rfloor, y^{\lceil * \rceil} : \dot{y} \Vdash \beta$
$\equiv \dot{\beta}^{\square} : *, \beta^{\lceil \square \rceil} : \dot{\beta} \to \lceil * \rceil, \dot{y}^{*} : \lfloor \beta \rfloor, y^{\lceil * \rceil} : \langle \beta \rangle \dot{y}$
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Adequacy - Example

If $\Gamma \vdash A : B : \lceil s \rceil$, then $\langle \Gamma \rangle \vdash \langle A \rangle : \lfloor A \rfloor \Vdash B : \lceil s \rceil$.

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Example

Take $\Gamma := \beta^{\lceil \Box \rceil} : \lceil * \rceil, y^{\lceil * \rceil} : \beta$, $A := y$, $B := \beta$, $s : \lceil * \rceil$. Then:
$\langle \beta^{\lceil \Box \rceil} : \lceil * \rceil, y^{\lceil * \rceil} : \beta \rangle \vdash \langle y \rangle : \lfloor y \rfloor \Vdash \beta : \lceil * \rceil$

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$\lceil * \rceil$
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Realizability increases arity of parametricity

Theorem

For any tuple terms $(B, \overline{C})$*,*

- $(B, \overline{C}) \in [\![A]\!]_{n+1} = B \Vdash \overline{C} \in [\![A]\!]_n$
- $(\!|A|\!)_{n+1} = \langle (\!|A|\!)_n \rangle$

Proof.

By induction on the structure of A. □

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Realizability increases arity of parametricity - Example

$$(B, \overline{C}) \in [\![A]\!]_{n+1} = B \Vdash \overline{C} \in [\![A]\!]_n$$

Example

$$(f, g) \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]_2 = f \Vdash g \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]$$

Realizability increases arity of parametricity - Example

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$(f, g) \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]_2 = f \Vdash g \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]$
We have: $(f, g) \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]_2$

Realizability increases arity of parametricity - Example

$$(B, \overline{C}) \in \llbracket A \rrbracket_{n+1} = B \Vdash \overline{C} \in \llbracket A \rrbracket_n$$

Example

$(f, g) \in \llbracket \Pi(\alpha : *).\alpha \to \alpha \rrbracket_2 = f \Vdash g \in \llbracket \Pi(\alpha : *).\alpha \to \alpha \rrbracket$
We have: $(f, g) \in \llbracket \Pi(\alpha : *).\alpha \to \alpha \rrbracket_2$
$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *).\Pi(\mathring{\alpha} : (\alpha_1, \alpha_2) \in \llbracket * \rrbracket).(f\alpha_1, g\alpha_2) \in \llbracket \Pi x : \alpha.\alpha \rrbracket$

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$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *)(\mathring{\alpha} : (\alpha_1, \alpha_2) \in [\![*]\!])(x_1 : \alpha_1)(x_2 : \alpha_2)(\mathring{x} :$
$(x_1, x_2) \in [\![\alpha]\!]).(f\alpha_1 x_1, g\alpha_2 x_2) \in [\![\alpha]\!]$

Realizability increases arity of parametricity - Example

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$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *)(\mathring{\alpha} : (\alpha_1, \alpha_2) \in [\![*]\!])(x_1 : \alpha_1)(x_2 : \alpha_2)(\mathring{x} : (x_1, x_2) \in [\![\alpha]\!]).(f\alpha_1 x_1, g\alpha_2 x_2) \in [\![\alpha]\!]$
$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *)(\mathring{\alpha} : \alpha_1 \to \alpha_2 \to \lceil * \rceil)(x_1 : \alpha_1)(x_2 : \alpha_2)(\mathring{x} : \mathring{\alpha}\, x_1\, x_2).\mathring{\alpha}(f\alpha_1 x_1)(g\alpha_2 x_2)$

Realizability increases arity of parametricity - Example

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$(f, g) \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]_2 = f \Vdash g \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]$
We have: $(f, g) \in [\![\Pi(\alpha : *).\alpha \to \alpha]\!]_2$
$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *).\Pi(\mathring{\alpha} : (\alpha_1, \alpha_2) \in [\![*]\!]).(f\alpha_1, g\alpha_2) \in [\![\Pi x : \alpha.\alpha]\!]$
$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *)(\mathring{\alpha} : (\alpha_1, \alpha_2) \in [\![*]\!])(x_1 : \alpha_1)(x_2 : \alpha_2)(\mathring{x} :$
$(x_1, x_2) \in [\![\alpha]\!]).(f\alpha_1 x_1, g\alpha_2 x_2) \in [\![\alpha]\!]$
$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *)(\mathring{\alpha} : \alpha_1 \to \alpha_2 \to \lceil * \rceil)(x_1 : \alpha_1)(x_2 : \alpha_2)(\mathring{x} :$
$\mathring{\alpha}\, x_1\, x_2).\mathring{\alpha}(f\alpha_1 x_1)(g\alpha_2 x_2)$
Working out the R.H.S.:
$f \Vdash \Pi(\alpha_1 : *)(\mathring{\alpha} : \alpha_1 \to \lceil * \rceil)(x_1 : \alpha_1)(\mathring{x} : \mathring{\alpha}\, x_1).\mathring{\alpha}(g\alpha_1 x_1)$
$\equiv \Pi(\alpha_1 : *)(\alpha_2 : *)(\mathring{\alpha} : \alpha_1 \to \alpha_2 \to \lceil * \rceil)(x_1 : \alpha_1)(x_2 : \alpha_2)$
$(\mathring{x} : \mathring{\alpha}\, x_1\, x_2).\mathring{\alpha}(g\, \alpha_1\, x_1)(f\, \alpha_2\, x_2)$

Realizability to Parametricity

Using the theorem:

Theorem

For any tuple terms $(B, \overline{C})$,

- $(B, \overline{C}) \in [\![A]\!]_{n+1} = B \Vdash \overline{C} \in [\![A]\!]$
- $(\!|A|\!)_{n+1} = \langle (\!|A|\!)_n \rangle$

We can prove:

Corollary

- $\overline{C} \in [\![A]\!]_n = C_1 \Vdash C_2 \Vdash \ldots \Vdash C_n \Vdash \lceil A \rceil$
- $(\!|A|\!)_n = \langle \ldots \langle \lceil A \rceil \rangle \ldots \rangle$

Proof.

By induction on n, using $(\!|A|\!)_0 = \lceil A \rceil$ for the base case. □

Third level

Definition (Third-level system)

Given a PTS $P = (\mathcal{S}, \mathcal{A}, \mathcal{R})$, define $P^3 = (P^2)^2$, where the sort-lifting $\lceil \cdot \rceil$ used by both instances of the $\cdot^2$ translation is the same.

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That is, $P^3 = (\mathcal{S}^3, \mathcal{A}^3, \mathcal{R}^3)$, where

$$
\begin{aligned}
\mathcal{S}^3 &= \mathcal{S} \cup \lceil \mathcal{S} \rceil \cup \lceil \lceil \mathcal{S} \rceil \rceil \\
\mathcal{A}^3 &= \mathcal{A} \cup \lceil \mathcal{A} \rceil \cup \lceil \lceil \mathcal{A} \rceil \rceil \\
\mathcal{R}^3 &= \mathcal{R} \cup \lceil \mathcal{R} \rceil \cup \lceil \lceil \mathcal{R} \rceil \rceil \\
&\qquad \cup \{(s_1, \lceil s_3 \rceil, \lceil s_3 \rceil), (\lceil s_1 \rceil, \lceil \lceil s_3 \rceil \rceil, \lceil \lceil s_3 \rceil \rceil) \mid (s_1, s_2, s_3) \in \mathcal{R}\} \\
&\qquad \cup \{(s_1, \lceil s_2 \rceil, \lceil s_2 \rceil), (\lceil s_1 \rceil, \lceil \lceil s_2 \rceil \rceil, \lceil \lceil s_2 \rceil \rceil) \mid (s_1, s_2) \in \mathcal{A}\}
\end{aligned}
$$

Extensions of parametricity and projection

- Extend parametricity ($C \in \llbracket T \rrbracket$) to map second-level constructs in P^2 to third-level constructs in P^3.
- Extend projection ($\lfloor \cdot \rfloor$) to map third-level constructs to second level

Parametricity to realizability

Lemma

If A is a first-level term, then

$$A = \lfloor C \in [\![A]\!]_1 \rfloor \qquad \textit{and} \qquad \langle A \rangle = \lfloor (\!|A|\!)_1 \rfloor.$$

Proof.

By induction on the structure of A. $\square$

Parametricity to realizability

Lemma

If A is a first-level term, then

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Proof.

By induction on the structure of A. $\square$

Theorem (From parametricity to realizability)

If A is a second-level term, then

$$C \Vdash A = \lfloor \lceil C \rceil \in [\![A]\!]_1 \rfloor \qquad \textit{and} \qquad \langle A \rangle = \lfloor (\!|A|\!)_1 \rfloor$$

Proof.

By induction on the structure of A, using the above lemma. $\square$

From parametricity to realizability - Example

Theorem (From parametricity to realizability)

If A is a second-level term, then

$$C \Vdash A = \lfloor \lceil C \rceil \in [\![A]\!]_1 \rfloor \qquad \textit{and} \qquad \langle A \rangle = \lfloor [\![A]\!]_1 \rfloor$$

Example

Consider $A = \mathring{\alpha} \to \mathring{\alpha}$. Then
$\lfloor \lceil C \rceil \in [\![\mathring{\alpha} \to \mathring{\alpha}]\!]_1 \rfloor$
$\equiv \lfloor \Pi(\mathring{x} : \mathring{\alpha}).\mathring{\mathring{\alpha}}\mathring{x} \to \mathring{\mathring{\alpha}}(\lceil C \rceil \mathring{x}) \rfloor$
$\equiv \Pi(x : \alpha).\mathring{\alpha}x \to \mathring{\alpha}(Cx)$
while
$C \Vdash \mathring{\alpha} \to \mathring{\alpha}$
$\equiv \Pi(x : \alpha)(\mathring{x} : x \Vdash \mathring{\alpha}).\mathring{\alpha}(Cx)$
$\equiv \Pi(x : \alpha).\mathring{\alpha}x \to \mathring{\alpha}(Cx)$

Conclusion

- PTS P

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- P^2

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- From realizability to parametricity: n-ary parametricity is the composition of lifting and n realizability steps

$$\overline{C} \in [\![A]\!]_n = C_1 \Vdash C_2 \Vdash \ldots \Vdash C_n \Vdash \lceil A \rceil$$

Conclusion

- PTS P
- P^2
- From realizability to parametricity: n-ary parametricity is the composition of lifting and n realizability steps

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Conclusion

- PTS P
- P^2
- From realizability to parametricity: n-ary parametricity is the composition of lifting and n realizability steps

$$\overline{C} \in [\![A]\!]_n = C_1 \Vdash C_2 \Vdash \ldots \Vdash C_n \Vdash \lceil A \rceil$$

- P^3
- From parametricity to realizability: realizability is the composition of parametricity and projection

$$C \Vdash A = \lfloor \lceil C \rceil \in [\![A]\!]_1 \rfloor$$

Q & A