Interpretation of Universes (*Prop*, $Type_j$), Propositions and Π/Σ-types

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December 3, 2025

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ω-sets

Definition

- $A = (|A|, \Vdash_A)$
- Where $|A|$: carrier set
- $\Vdash_A$: realizability relation $(\omega \times |\, A\,|)$
- $n \Vdash a$: n implements a

How are ω-sets used?

Definition ω-set

- $A = (|A|, \Vdash_A)$
- Where $|A|$ is called the carrier set,
  and $\Vdash_A$ is called the realizability relation.

In practice

- Context: $[\![\Gamma]\!] : \omega{-}\mathbf{Set}$
- Type: $[\![\Gamma \vdash A : \mathit{Type}_j]\!] : [\![\Gamma]\!] \to \omega{-}\mathbf{Set}$
- $\sigma(\Gamma, A)$: expand Γ with type A

Requirements of Type universes

How can we interpret a type universe, such that:

- $Prop \in Type_0 \in Type_1 \in Type_2 \in \ldots$
- $Prop \subseteq Type_0 \subseteq Type_1 \subseteq Type_2 \subseteq \ldots$
- $Type_j$ is closed under Σ and Π (predicatively)
- $Prop$ is closed under Π (impredicatively for arbitrary products).

Interpreting and extending the context

Definition $\sigma(\Gamma, A)$

Let Γ be an ω-set and A be a $|\Gamma|$ indexed ω-set

$$|\sigma(\Gamma, A)| =_{\text{df}} \{(\gamma, a) \mid \gamma \in |\Gamma|, a \in |A(\gamma)|\}$$

$$\langle m, n\rangle \Vdash_{\sigma(\Gamma,a)} (\gamma, a) \text{ if and only if } m \Vdash_{\Gamma} \gamma \text{ and } n \Vdash_{A(\gamma)} a$$

- Extend context Γ with A
- $[\![\Gamma, x : A]\!] =_{\text{df}} \sigma([\![\Gamma]\!], [\![\Gamma \vdash A : \mathit{Type}_j]\!])$

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Again: Requirements of Type universes

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Π/Σ types

Definition Σ-types

- $\Sigma x : A.B$
- Dependent pair type
- Type of the second element of the pair depends on value of the first element

Definition Π-types

- $\Pi x : A.B$
- Dependent function type
- Type of the result of the function depends on value of x.

Σ-Types as ω-sets

Definition $\sigma_\Gamma(A, B)$

- Let $A : \Gamma \to \omega{-}\mathbf{Set}$ and $B : |\,\sigma(\Gamma, A)\,| \to \omega{-}\mathbf{Set}$

$$\sigma_\Gamma(A, B) : |\,\Gamma\,| \to \omega{-}\mathbf{Set}$$

The carrier set is defined by:

$$|\,\sigma_\gamma(A, B)(\gamma)\,| =_{\mathrm{df}} \{(a, b) \mid a \in |\,A(\gamma)\,|\,, b \in |\,B(\gamma, a)\,|\}$$

Realizability relation is defined by:

$$\langle m, n\rangle \Vdash_{\sigma_\Gamma(A,B)(\gamma)} (a, b) \text{ if and only if } m \Vdash_{A(\gamma)} a \text{ and } n \Vdash_{B(\gamma,a)} b$$

Π-Types as ω-sets

Definition $\pi_\Gamma(A, B)$

- Let $A : \Gamma \to \omega\text{−}\mathbf{Set}$ and $B : |\,\sigma(\Gamma, A)\,| \to \omega\text{−}\mathbf{Set}$

$$\pi_\Gamma(A, B) : |\,\Gamma\,| \to \omega\text{−}\mathbf{Set}$$

$$|\,\pi_\Gamma(A, B)(\gamma)\,| =_{\mathrm{df}} \left\{ f \in \prod_{a \in |\,A(\gamma)\,|} |\,B(\gamma, a)\,| \;\middle|\; \exists n \in \omega.\ n \Vdash_{\pi_\Gamma(A,B)(\gamma)} f \right\}$$

$$n \Vdash_{\pi_\Gamma(A,B)(\gamma)} \text{ iff } \forall a \in |\,A(\gamma)\,| :$$

$$\forall m \in \omega.\ m \Vdash_{|\,A(\gamma)\,|} a \implies nm \Vdash_{|\,B(\gamma,a)\,|} f(a)$$

Interpretation of Σ and Π Types

- Let $A : \Gamma \to \omega{-}\mathbf{Set}$ and $B : |\,\sigma(\Gamma, A)\,| \to \omega{-}\mathbf{Set}$
  - $\sigma_\Gamma(A, B) : |\,\Gamma\,| \to \omega{-}\mathbf{Set}$
  - $\pi_\Gamma(A, B) : |\,\Gamma\,| \to \omega{-}\mathbf{Set}$
- $[\![\Gamma \vdash \Sigma x : A.B : Type_j]\!] =_{\text{df}} \sigma_{[\![\Gamma]\!]}[\![\Gamma \vdash A : Type_j]\!], [\![\Gamma, x : A \vdash B : Type_j]\!])$
- $[\![\Gamma \vdash \Pi x : A.B : Type_j]\!] =_{\text{df}} \pi_{[\![\Gamma]\!]}[\![\Gamma \vdash A : Type_j]\!], [\![\Gamma, x : A \vdash B : Type_j]\!])$

Again: Requirements of Type universes

How can we interpret a type universe, such that:

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Cumulative hierarchy of set

- $Prop \in Type_0 \in Type_1 \in Type_2 \in \dots$
- $Prop \subseteq Type_0 \subseteq Type_1 \subseteq Type_2 \subseteq \dots$

$$Prop = B \cup N$$

$$\downarrow$$

$$Type_j = \dots$$

Cumulative hierarchy of set

- $Prop \in Type_0 \in Type_1 \in Type_2 \in \ldots$
- $Prop \subseteq Type_0 \subseteq Type_1 \subseteq Type_2 \subseteq \ldots$

$$Prop = \mathbb{B} \cup \mathbb{N}$$

$$\downarrow$$

$$Type_0 = \mathbb{B} \cup \mathbb{N} \cup \mathcal{P}(\mathbb{B} \cup \mathbb{N})$$

$$Type_1 = \mathbb{B} \cup \mathbb{N} \cup \mathcal{P}(\mathbb{B} \cup \mathbb{N}) \cup \mathcal{P}(\mathcal{P}(\mathbb{B} \cup \mathbb{N}))$$

Defining $\omega-\mathbf{Set}(j)$

- $\kappa_1 < \kappa_2 < \kappa_3 < \dots$
- V_{κ_j}: set universe
- $\omega-\mathbf{Set}(j)$ subcategory of $\omega-\mathbf{Set}$
- $Type_j$ corresponds to category $\omega-\mathbf{Set}(j)$
- All carrier sets in $\omega - \mathbf{Set}(j)$ are in universe V_{κ_j}

Interpretation of universe $Type_j$

- $\Delta(Obj(\omega - \mathbf{Set}(j))) \in Obj(\omega - \mathbf{Set}(j+1))$
- $[\![\Gamma \vdash Type_j : Type_{j+1}]\!](\gamma) =_{\mathsf{df}} \Delta(Obj(\omega - \mathbf{Set}(j)))$

where $\Delta : \mathbf{Set} \to \omega - \mathbf{Set}$

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Modest Sets

Definition **M**

$$\forall n \in \omega \; \forall a, b \in |A| .\; n \Vdash_A a \textit{ and } n \Vdash_A b \implies a = b$$

- **M** is too "big"
- Define "smaller" category **PROP**

PROP

Definition $Obj(\textbf{PROP})$

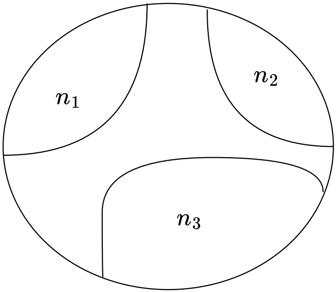
Let R be a partial equivalence relation

$$Obj(\textbf{PROP}) =_{\text{df}} \{(Q(R), \in) \mid R \subseteq \omega \times \omega\}$$

- Elements in $Type_j$ are ω-sets
- Elements in $Prop$ are also ω-set
- Should be small

Quotient Set with Respect to R

- Partial equivalence relation: $R \subseteq \omega \times \omega$
- $[n_1]_R$: $\{m \in \omega \mid n_1 R m\}$
- All implementation related to n_1
- $Q(R) = \{[n]_R \mid (n, n) \in R\}$



Definition of **back**

Lemma

There is an equivalence of categories $\mathbf{back} : \mathbf{M} \to \mathbf{PROP}$ *such that:*

- $\mathbf{back}(A) \cong A$ *for* $A \in Obj(\mathbf{M})$
- $\mathbf{back}(P) = P$ *for* $P \in Obj(\mathbf{Prop})$

Definition $\mathbf{back} : \mathbf{M} \to \mathbf{PROP}$

For $A \in Obj(\mathbf{M})$

$$\mathbf{back}(A) =_{\mathrm{df}} (Q(R_A), \in)$$

where

$$R_A = \{(n, m) \mid \exists a \in A.\ n \Vdash_A a \text{ and } m \Vdash_A a\}$$

Definition of **back** continued

- A morphism between ω-sets A and B is an $f : |A| \to |B|$ s.t.:

$$\exists n \in \omega \ \forall a \in |A| \ \forall m \in \omega. \ m \Vdash_A a \implies nm \Vdash_B f(a)$$

- A morphism is an ω-set

Definition **back** on morphisms

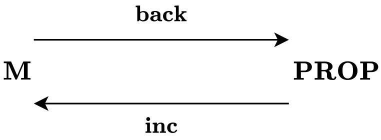
$\mathbf{back} : \mathbf{M} \to \mathbf{PROP}$
Let $f : |A| \to |B|$ in $\mathbf{M}$
$\mathbf{back}(f) : |A| \to |B|$ in $\mathbf{PROP}$

$$\mathbf{back}(f)([p]_{R_A}) \quad =_{df} \quad [np]_{R_B} \text{ where } n \Vdash_{A,B} f$$

Definition of **inc**

Definition of **inc**

- Let **inc** :**PROP** $\rightarrow$ **M**, such that it is the inverse of **back**
- It is the inclusion functor from **PROP** to **M**
- $$id : id_{\mathbf{PROP}} \rightarrow \mathbf{back} \circ \mathbf{inc}$$
  $$\eta : id_{\mathbf{M}} \rightarrow \mathbf{inc} \circ \mathbf{back}$$



Proof lemma part 1

Lemma (part 1)

There is an equivalence of categories $\mathbf{back} : \mathbf{M} \to \mathbf{PROP}$ *s.t:*

- $\mathbf{back}(A) \cong A$ *for* $A \in Obj(\mathbf{M})$

Proof.

- Let $A \in Obj(A)$ and $a \in |\, A\,|$
- $\eta_A(a) =_{\text{df}} [n]_{R_A}$ where $n \Vdash_A a$
- $\mathbf{back}(A) = (Q(R_A), \in) = \mathbf{inc} \circ \mathbf{back}(A) \cong A$

□

Proof lemma part 2

Lemma (part 2)

There is an equivalence of categories $\mathbf{back} : \mathbf{M} \to \mathbf{PROP}$ *s.t.:*

- $\mathbf{back}(P) = P$ *for* $P \in Obj(\mathbf{Prop})$

Proof.

Let $P = (Q(R), \in) \in Obj(\mathbf{PROP})$
$\mathbf{back}(P) = (Q(R_P), \in) = (Q(R), \in) = P$

$$\begin{aligned} R_p &= \{(n,m) \mid \exists a \in |P| \,.\, n \Vdash_P a \text{ and } m \Vdash_P a\} \\ &= \{(n,m) \mid \exists [a]_R \in Q(R).\ m \Vdash_P [a]_R \text{ and } n \Vdash_P [a]_R\} \\ &= \{(n,m) \mid \exists a \in \omega.\ (m, n \in [a]_R)\} \\ &= R \end{aligned}$$

□

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Requirement 4 of Type universes

- *Prop* is closed under Π
- First use $\pi_{[\![\Gamma]\!]}$ to form the product.
  Then use **back** to take it back into a family of objects in **PROP**.

$$[\![\Gamma \Vdash \Pi x : A.P : Prop]\!] =_{df} \mathbf{back} \circ \pi_{[\![\Gamma]\!]}([\![\Gamma \Vdash A : T_\Gamma(A)]\!], [\![\Gamma, x : A \Vdash P : Prop]\!])$$

Interpretation universe **PROP**

Based on these, we interpret the universe *Prop* as the following $|\llbracket\Gamma\rrbracket|$-indexed family of ω-sets, for $\gamma \in |\llbracket\Gamma\rrbracket|$,

$$\llbracket\Gamma \vdash \mathit{Prop} : \mathit{Type}_0\rrbracket(y) =_{df} \Delta(\mathit{Obj}(\mathbf{PROP}))$$

Similar to how the universes *Type* are defined:

$$\llbracket\Gamma \vdash \mathit{Type}_j : \mathit{Type}_{j+1}\rrbracket(\gamma) =_{\mathrm{df}} \Delta(\mathit{Obj}(\omega{-}\mathbf{Set}(j)))$$

where $\Delta : \mathbf{Set} \to \omega - \mathbf{Set}$