Induction Is Not Derivable in Second Order Dependent Type Theory

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What do we want to prove?

- Induction is not derivable in $\lambda P2$.
- So, for example, with

$$\text{ind} := \Pi P : \text{nat} \to \star.(PO) \to (\Pi y : \text{nat}.(Py) \to (P(\text{succ } y))) \\ \to \Pi x : \text{nat}.(Px)$$

- we prove that for any context Γ and pseudo-term N:

$$\Gamma \nvdash \text{ind} : N$$

What do we need to prove it?

- Introduce notation, definitions and lemmas.
- Introduce counter-model.

> **Note** The counter-model will be a model of $\lambda P2$, for which the induction property is not valid.

Validity

Definition 10. For $\mathcal{M}$ a $\lambda P2$-model, Γ a context, σ a type in Γ and ξ, ρ valuations such that $\xi, \rho \models \Gamma$, we say that σ is *valid in* $\mathcal{M}$ *under* ξ, ρ, notation

$$\mathcal{M}, \xi, \rho \Vdash^{\lambda P2} \sigma,$$

if

$$[\![\sigma]\!]^{\mathcal{M}}_{\xi\rho} \neq \emptyset.$$

So, to prove the non-derivability of ind in $\lambda P2$, we are looking for a $\lambda P2$-model $\mathcal{M}$ such that

$$\mathcal{M} \not\Vdash^{\lambda P2} \text{ind}.$$

Consistency

Definition 11. A $\lambda P2$-model M is *consistent* if $\emptyset \in \mathcal{P}$. For a $\lambda P2$-model, being consistent is equivalent to saying that $[\![\bot]\!] = \emptyset$, because $[\![\bot]\!]$ is the minimal element (w.r.t. $\subseteq$) of $\mathcal{P}$. Here, $\bot$ is defined as usual as $\Pi\alpha : \star.\alpha$.
Note that the polyset structures of Example 2 all yield a consistent $\lambda P2$-model.

Convention 12. From now on we only discuss consistent $\lambda P2$-models.

Connectives

Definition 13. In a $\lambda P2$-model $\mathcal{M} = \langle \mathcal{A}, \mathcal{P}, \mathcal{N} \rangle$ we define the 'connectives' $\bot$, $\neg$, $\wedge$, $\vee$ and $\exists$ as follows. ($X, Y \in \mathcal{P}$, $F : X \to \mathcal{P}$ and $Y_i \in \mathcal{P}$ for all $i \in I$; as in types, we let brackets associate to the right.)

$$\bot := \bigcap_{Z \in P} Z, \qquad \neg X := X \to \bot,$$

$$X \wedge Y := \bigcap_{Z \in P} (X \to Y \to Z) \to Z,$$

$$X \vee Y := \bigcap_{Z \in P} (X \to Z) \to (Y \to Z) \to Z,$$

$$\exists_{x \in X} F(x) := \bigcap_{Z \in P} (\Pi x \in X.\, F(x) \to Z) \to Z,$$

$$\exists_{i \in I} Y_i := \bigcap_{Z \in P} (\bigcap_{i \in I} Y_i \to Z) \to Z.$$

Logic in λP2-models

Lemma 1. The following holds in arbitrary (consistent) $\lambda P2$-models $\mathcal{M}$:

$$\neg X = \emptyset \;\Leftrightarrow\; X \neq \emptyset, \tag{1}$$

$$X \to Y \neq \emptyset \;\Leftrightarrow\; \text{if } X \neq \emptyset \text{ then } Y \neq \emptyset, \tag{2}$$

$$X \wedge Y \neq \emptyset \;\Leftrightarrow\; X \neq \emptyset \text{ and } Y \neq \emptyset, \tag{3}$$

$$X \vee Y \neq \emptyset \;\Leftrightarrow\; X \neq \emptyset \text{ or } Y \neq \emptyset, \tag{4}$$

$$\exists_{x \in X}\, F(x) \neq \emptyset \;\Leftrightarrow\; \exists t \in X\ (F(t) \neq \emptyset), \tag{5}$$

$$\exists_{i \in I}\, Y_i \neq \emptyset \;\Leftrightarrow\; \exists i \in I\ (Y_i \neq \emptyset), \tag{6}$$

$$\Pi_{x \in X} F(x) \neq \emptyset \;\Rightarrow\; \forall t \in X\ (F(t) \neq \emptyset), \tag{7}$$

$$\bigcap_{i \in I} Y_i \neq \emptyset \;\Rightarrow\; \forall i \in I\ (Y_i \neq \emptyset). \tag{8}$$

Logic in λP2-models

Lemma 2. For a simple $\lambda P2$-model over $\mathcal{A}$ the reverse implications in Lemma 1, cases (7) and (8), hold. Similarly for a $\lambda P2$-model generated from a set C.

Reminder A simple $\lambda P2$ model over $\mathcal{A}$ is a model made with the simple polyset structure over $\mathcal{A}$, so $\mathcal{P} = \{\emptyset, \mathbf{A}\}$.

$\lambda P2$-model generated from a set C:

$$\mathcal{P} := \{\, X \subseteq \Lambda(C) \mid X = \emptyset \ \vee \ C \subseteq X \,\}.$$

For (7), if for all $t \in X$, $F_t \neq \emptyset$, then there is an element q such that

$$\forall t \in X \ (q \in F(t))$$

and hence $\lambda^* x.\, q \in \Pi_{t \in X} F(t)$. Case (8) is immediate.

Classical Logic in λP2-models

Lemma 3. All $\lambda P2$-models satisfy classical logic, i.e.

$$\neg\neg X \to X \neq \emptyset$$

for all $X \in \mathcal{P}$ in all $\lambda P2$-models.

Proof. We reason classically in the models, using Lemma 1. Let $X \in \mathcal{P}$. If $X \neq \emptyset$, say $t \in X$, then $\neg\neg X \to X \neq \emptyset$, because e.g. $\lambda^* x.t \in \neg\neg X \to X$. If $X = \emptyset$, then $\neg X = \mathbf{A}$, so $\neg\neg X = \emptyset$, so $\neg\neg X \to X = \mathbf{A}$

Reminder Lemma 1 is the logic in $\lambda P2$-models.

Leibniz equality

Equality is defined in $\lambda P2$ using Leibniz equality: for $\sigma : \star, M, N : \sigma$

$$M =_\sigma N := \Pi P : \sigma \to \star.(PM) \to (PN)$$

Model intepretation

Lemma 4. Given a $\lambda P2$-model $\mathcal{M}$, a type σ and terms $M, N : \sigma$, we have

$$\mathcal{M}, \xi, \rho \Vdash M =_\sigma N \Leftrightarrow (\![M]\!)_\rho = (\![N]\!)_\rho.$$

Note: A term "M is Leibniz equal to N"is valid in the model if and only if M and N have the same interpretations.

Model intepretation cont.

Lemma 4. Given a $\lambda P2$-model $\mathcal{M}$, a type σ and terms $M, N : \sigma$, we have

$$\mathcal{M}, \xi, \rho \Vdash M =_\sigma N \Leftrightarrow (\![M]\!)_\rho = (\![N]\!)_\rho.$$

Proof. $\Rightarrow$:
Step 1: Rewrite $M =_\sigma N$ to

$$\bigcap_{Q \in [\![\sigma]\!] \to \mathcal{P}} Q(\![M]\!)_\rho \to Q(\![N]\!)_\rho \neq \emptyset$$

Step 2: Define a function Q such that $Qx \neq \emptyset$ iff $x = (\![M]\!)_\rho$.
Step 3: Then, $Q(\![N]\!)_\rho \neq \emptyset$.
Conclusion: So, $(\![M]\!)_\rho = (\![N]\!)_\rho$.

Model intepretation cont.

Lemma 4. Given a $\lambda P2$-model $\mathcal{M}$, a type σ and terms $M, N : \sigma$, we have

$$\mathcal{M}, \xi, \rho \Vdash M =_\sigma N \Leftrightarrow (\![M]\!)_\rho = (\![N]\!)_\rho.$$

Proof. $\Leftarrow$:
Step 1: Since $(\![M]\!)_\rho = (\![N]\!)_\rho$,

$$Q(\![M]\!)_\rho = Q(\![N]\!)_\rho$$

Step 2: thus

$$\lambda^* x.x \in \bigcap_{Q \in [\![\sigma]\!] \to \mathcal{P}} Q(\![M]\!)_\rho \to Q(\![N]\!)_\rho$$

Conclusion: So $M =_\sigma N$ is valid in the model.

Induction defintion

Definition 14. Given a closed $\lambda P2$-type N and closed terms $0 : N$ and $S : N \to N$, we define the type $\mathrm{ind}_{N,0,S}$ by

$$\Pi P : N \to \star . P0 \to (\Pi x : N . Px \to P(Sx)) \to \Pi x : N . Px.$$

Induction in λP2-models

Lemma 5. For a $\mathcal{M} = \langle \mathcal{A}, \mathcal{P}, \mathcal{N} \rangle$ a $\lambda P2$-model,

$$\mathcal{M} \Vdash \mathrm{ind}_{N,0,S} \Rightarrow [\![N]\!] = \{S^n 0 | n \in \mathbb{N}\}$$

If, moreover, the test-for-zero and predecessor function are definable on the type N in the model $\mathcal{M}$, then also

$$[\![N]\!] = \{S^n 0 | n \in \mathbb{N}\} \Rightarrow \mathcal{M} \Vdash \mathrm{ind}_{N,0,S}$$

The interpretation of N is the natural numbers if (and only if) $\mathrm{ind}_{N,0,S}$ is valid in $\mathcal{M}$.

Induction in λP2-models cont.

Lemma 5. For a $\mathcal{M} = \langle \mathcal{A}, \mathcal{P}, \mathcal{N} \rangle$ a $\lambda P2$-model,

$$\mathcal{M} \Vdash \mathrm{ind}_{N,0,S} \Rightarrow [\![N]\!] = \{S^n 0 | n \in \mathbb{N}\}$$

Proof.
Step 1: We re-write $\mathcal{M} \Vdash \mathrm{ind}_{N,0,S}$ as

$$\bigcap_{Q \in N \to \mathcal{P}} Q0 \to (\Pi_{t \in N} Qt \to Q(St)) \to \Pi_{t \in N} Qt \neq \emptyset.$$

Step 2: Let X be some non-empty element of $\mathcal{P}$. Define a function $Q : N \to \mathcal{P}$ such that if $t = S^n 0$, then $Qt = X$ else $Qt = \emptyset$.
Step 3: By definition, $Q0 \neq \emptyset$, and $\Pi_{t \in N} Qt \to Q(St) \neq \emptyset$.
Step 4: Thus, $\Pi_{t \in N} Qt \neq \emptyset$, so an element $M \in \Pi_{t \in N} Qt$.

Induction in λP2-models cont.

Lemma 5. For a $\mathcal{M} = \langle \mathcal{A}, \mathcal{P}, \mathcal{N} \rangle$ a $\lambda P2$-model,

$$\mathcal{M} \Vdash \mathrm{ind}_{N,0,S} \Rightarrow [\![N]\!] = \{S^n 0 | n \in \mathbb{N}\}$$

Proof cont.
Step 5: Now, suppose $q \in N$ with $q \neq S^n 0$. Then, by definition of Q, $Qq = \emptyset$.
Step 6: Also, $Mq \in Qq$, with $Mq \neq \emptyset$.
Conclusion: Contradiction, so all $q \in N$ are of the form $S^n 0$.

Induction in λP2-models cont.

Lemma 5 cont. If, moreover, the test-for-zero and predecessor function are definable on the type N in the model $\mathcal{M}$, then also

$$[\![N]\!] = \{S^n 0 | n \in \mathbb{N}\} \Rightarrow \mathcal{M} \Vdash \mathrm{ind}_{N,0,S}$$

Proof.

Step 1: We suppose test-for-zero and the predecessor function are definable in the model.

Step 2: Suppose $N = \{S^n 0 | n \in \mathbb{N}\}$ (the assumption).

Step 3: Again, re-write $\mathcal{M} \Vdash \mathrm{ind}_{N,0,S}$ as

$$\bigcap_{Q \in N \to \mathcal{P}} Q0 \to (\Pi_{t \in N} Qt \to Q(St)) \to \Pi_{t \in N} Qt \neq \emptyset.$$

This is our goal.

Step 4: Define a function $Q \in N \to \mathcal{P}$ arbitrarily, and let $Z \in Q0$.

Step 5: Define another function $F \in \Pi_{t \in N} Qt \to Q(St)$.

Step 6: We are looking for an element of $\Pi_{t \in N} Qt$.

Induction in λP2-models cont.

Lemma 5 cont. If, moreover, the test-for-zero and predecessor function are definable on the type N in the model $\mathcal{M}$, then also

$$[\![N]\!] = \{S^n 0 | n \in \mathbb{N}\} \Rightarrow \mathcal{M} \Vdash \mathrm{ind}_{N,0,S}$$

Proof cont.
Step 7: This element is given by an H which is the solution to

$$Hx = \text{if } \mathrm{Zero}(x) \text{ then } Z \text{ else } F(x-1)(H(x-1)).$$

Step 8: We can obtain the element by taking a fixed point of

$$\lambda^* hx.\text{if } \mathrm{Zero}(x) \text{ then } Z \text{ else } F(x-1)(H(x-1))$$

Note: We need the test-for-zero and predecessor to be able to define this in H.

Proof

Theorem 2. Induction over the natural numbers is not derivable in $\lambda P2$ for any type N and terms $0 : N$ and $S : N \to N$.

Recap: simple $\lambda P2$-model over Λ

Example 1.
1. A standard example of a weca is Λ, consisting of the classes of open λ-terms modulo β-equality. Thus, $\mathbf{A}$ is just Λ/β and $[M] = [N]$ iff $M =_\beta N$. It is easily verified that this yields a weca.

Example 3.
2. The simple $\lambda P2$-model over $\mathcal{A}$ is $\mathcal{M} = \langle \mathcal{A}, \mathcal{P}, \mathcal{N} \rangle$, where $\mathcal{P}$ is the simple polyset structure over $\mathcal{A}$ (so $\mathcal{P} = \{\emptyset, \mathcal{A}\}$).

So, simple $\lambda P2$-model over Λ:
$\mathcal{M} = \langle \Lambda, \mathcal{P}, \mathcal{N} \rangle$, where $\mathcal{P} = \{\emptyset, \Lambda\}$.

Recap: interpretation functions

Definition 4 ...
We now define three interpretation functions:

$$\mathcal{V}(-) : \text{kinds} \to \mathcal{N}, \qquad [\![-]\!] : \begin{cases} \text{constructors} \to \bigcup \mathcal{N}, \\ \text{types} \to \mathcal{P}. \end{cases}$$

...

So for all types σ, $[\![\sigma]\!] \in \mathcal{P}$.

Assume induction is derivable

$$\mathcal{M} \Vdash \text{ind}_{N,0,S} \Rightarrow [\![N]\!] = \{S^n 0 | n \in \mathbb{N}\}$$

$$|\{S^n 0 | n \in \mathbb{N}\}| > 0$$

$$|[\![N]\!]| > 0$$

Example 3: $\mathcal{P} = \{\emptyset, \Lambda\}$
Definition 4: $[\![N]\!] \in \mathcal{P}$
We can conclude that $[\![N]\!] = \Lambda$
Contradiction: $\Lambda = \{S^n 0 | n \in \mathbb{N}\}$

Results

The arguments of **Lemma 5** and **Theorem 2** also apply to other data types like lists and trees and even to a finite data type like the booleans. So, induction is not derivable for any data type.

Other results

One more non-derivability result in λP2, based on our models:

Lemma 6. There are closed types σ, τ and a relation $R : \sigma \to \tau \to \star$ in $\lambda P2$ for which the Axiom of Choice, $(\Pi x{:}\sigma.\exists y{:}\tau.Rxy) \to (\exists f{:}\sigma \to \tau.\Pi x{:}\sigma.Rx(fx))$, is not derivable.

Explanation.
Assumption: For every element in $x \in \sigma$, there exists an element $y \in \tau$ such that Rxy.
Conclusion: There exists a function $f : \sigma \to \tau$ such that for every element $x \in \sigma$, $Rx(fx)$.

Other results

Lemma 6. There are closed types σ, τ and a relation $R : \sigma \to \tau \to \star$ in $\lambda P2$ for which the Axiom of Choice, $(\Pi x{:}\sigma.\exists y{:}\tau.Rxy) \to (\exists f{:}\sigma \to \tau.\Pi x{:}\sigma.Rx(fx))$, is not derivable.

Proof.
Step 1: Take $\sigma = \tau = \mathrm{nat}$ and $Rxy := x \neq_{\mathrm{nat}} y$.
Step 2: Consider the simple $\lambda P2$-model over $\mathbf{A} = \Lambda$.
Step 3: Now $\mathcal{M} \models \Pi x{:}\sigma.\exists y{:}\tau.Rxy$, because this is equivalent to (using Lemmas 1 and 4) $\forall t \in \Lambda \exists q \in \Lambda (t \neq_\beta q)$.
Step 4: On the other hand, $\mathcal{M} \not\models \exists f{:}\sigma \to \tau.\Pi x{:}\sigma.Rx(fx)$, because this is equivalent to the statement $\exists g \in \Lambda \forall t \in \Lambda (gt \neq_\beta t)$, which is not possible, because every element of Λ has a fixed point.

Other results

Remark 4. It is in general not the case in $\lambda P2$ that the induction principle for one data type (say the natural numbers) implies the induction principle for another data type (say booleans).

Remark 4 continued

For a counterexample consider the context

$$\Gamma = N : \star,\ 0 : N,\ S : N \to N,\ h : \mathsf{ind}\ _{N,0,S}$$

and the $\lambda P2$-model $\langle \Lambda(C), \mathcal{P}, N \rangle$, where

$$C = \{\, S^n(0) \mid n \in \mathbb{N} \,\}$$

(so the $S^n(0)$ are considered as constants) and $\mathcal{P}$ is the polyset structure generated from C.
Now, take valuations ξ and ρ with

$$\xi(N) = C, \qquad \rho(0) = 0, \qquad \rho(S) = S, \qquad \rho(h) = \lambda^* zfx.\, 0.$$

Then $\rho(h) \in [\![\mathsf{ind}_{N,0,S}]\!]_{\xi,\rho}$:

$$\lambda^*\, zfx.\, 0 \ \in \bigcup_{Q \in C \to P} Q0 \to (\Pi_{t \in C} Qt \to Q(St)) \to \Pi_{t \in C} Qt,$$

because for $Q \in C \to P$, $Z \in Q0$, $G \in \Pi_{t \in C}(Qt \to Q(St))$ and $t \in C$, we find that $t = S^n(0)$ (by definition of C) and for all $n \in \mathbb{N}$ we have $Q(S^n(0)) \neq \emptyset$ (by induction on n, using Z and G), so $0 \in Qt$.

Remark 4 continued

We conclude that

$$\xi, \rho \Vdash \Gamma.$$

Thus,

$$M, \xi, \rho \Vdash \mathsf{ind}_{N,0,S}$$

On the other hand, for any closed type B (the 'booleans') with closed terms $T : B$ and $F : B$, we have

$$[\![B]\!] \supsetneq \{ [F], [T] \},$$

so induction over booleans is not valid.
Reminder:

$$\mathcal{P} := \{ X \subseteq \Lambda(C) \mid X = \emptyset \ \lor \ C \subseteq X \}.$$

Adding induction

One may wonder what happens with the counterexample in the proof of **Theorem 2** if we add induction over natural numbers to $\lambda P2$ as a primitive concept, together with the associated reduction rules.
We extend $\lambda P2$ with a type constant N and term constants $0 : N$, $S : N \to N$,

$$R : \Pi P : N \to \star.\ (P\,0) \to (\Pi y : N.\ P\,y \to P(Sy)) \to \Pi x : N.\ P\,x.$$

Furthermore, we add the reduction rules:

$$R\,P\,z\,f\,0 \ \to_r \ z$$

$$R\,P\,z\,f\,(Sx) \ \to_r \ f\,x\,(R\,P\,z\,f\,x).$$

Adding induction continued

To make a model of this extension of $\lambda P2$ we have to give an interpretation to the constants in such a way that the equality rule for R is preserved. For Λ (that we used in the counter-model of Theorem 2), this can be achieved by adding primitive constants 0, S and R to Λ, with the reduction rules

$$R\,z\,f\,0 \;\longrightarrow_r\; z \qquad \text{and} \qquad R\,z\,f\,(Sx) \;\longrightarrow_r\; f\,x\,(R\,z\,f\,x).$$

Let us denote this extension of λ-calculus (it is a weca) by Λ^+. (So we interpret 0 by 0, S by S and R by R.) Now consider the simple Λ^+-model determined by the polyset structure $\{\emptyset, \Lambda\}$ and notice that it is not a model of this $\lambda P2$ extension, because $\text{ind}_{N,0,S}$ is empty in this model (so we cannot interpret R).