Polymorphism is not set-theoretic (part 2)

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Recap

We defined $\mathbf{B}$, a type such that:

- $\mathbf{B}$ contains no free type variables.
- $B = S^{\#}\mathbf{B}$, the interpretation of $\mathbf{B}$, has at least 2 elements.

We also defined

$$\begin{aligned} \mathbf{P} &= \Pi s.(((s \to \mathbf{B}) \to \mathbf{B}) \to s) \to s, \\ P &= S^{\#}\mathbf{P}, \\ T(X) &= (X \to B) \to B. \end{aligned}$$

Recap

Finally, we found a function $H : TP \to P$ such that for any T-algebra (s, f), there is a function $\rho_{sf} : P \to s$ making the following diagram commute:

$$\begin{array}{ccc} TP & \xrightarrow{T\rho_{sf}} & Ts \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{\rho_{sf}} & s \end{array}$$

In other words: ρ_{sf} is a homomorphism of T-algebras.

Outline of the proof

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Idea: use Lambek's lemma and an initial T-algebra.

Problem: (P, H) is not quite an initial T-algebra.

Solution: Restrict P until ρ_{sf} is unique.

Step 1: parametricity

Lemma

If $\lambda 2$ has a set-theoretic model, then there is a T-algebra (P', H') such that for any T-algebra (s, f), there is a homomorphism $\rho'_{sf} : P' \to s$ with the property that any other homomorphism $\vartheta : P' \to s$ satisfies $\rho'_{sf} = \rho'_{P'H'}; \vartheta$.

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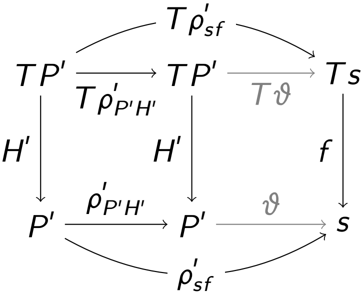
In other words, the following diagram commutes for any $\vartheta' : P' \to s$:



Step 1: parametricity

Definition (Parametricity)

We call $p \in P$ *parametric* if for any homomorphism α from (s, f) to (t, g) we have $\rho_{tg}(p) = \alpha(\rho_{sf}(p))$.

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- **Even more informally:** a parametric polymorphic function cannot do different things for different sets.

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- **Example:** $p[\sigma](f) = f(\varnothing[\sigma])$.

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- **Informally:** a parametric polymorphic function maps related values to related values.
- **Even more informally:** a parametric polymorphic function cannot do different things for different sets.
- **Example:** $p[\sigma](f) = f(\varnothing[\sigma])$.
- **Non-example:** $p[\sigma](f) = \begin{cases} 0 & \text{if } \sigma = \{0, 1\}, \\ f(\varnothing[\sigma]) & \text{otherwise.} \end{cases}$

Step 1: parametricity

Proof.

- Define $P' = \{p \in P \mid p \text{ is parametric}\}$.
- Let J be the inclusion $P' \to P$.
- If α is a homomorphism $(s, f) \to (t, g)$, then

$$J; \rho_{tg} = J; \rho_{sf}; \alpha$$

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- Define $P' = \{p \in P \mid p \text{ is parametric}\}$.
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- By Lemma 2, we have:

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- Thus, $H(TJ(h))$ is parametric for all $h \in TP'$.

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- Thus, $H(TJ(h))$ is parametric for all $h \in TP'$.
- Define H' as the *corestriction* of $TJ; H$.

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Now, the following diagram commutes:

$$\begin{array}{ccccc}
TP' & \xrightarrow{TJ} & TP & \xrightarrow{T\rho_{sf}} & Ts \\
\downarrow{\scriptstyle H'} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle f} \\
P' & \xrightarrow[J]{} & P & \xrightarrow[\rho_{sf}]{} & s
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- Thus, $\rho'_{sf} = J; \rho_{sf}$ is a homomorphism.
- Finally, let $\vartheta : P' \to s$ be another homomorphism.
- All elements $p \in P'$ are parametric, so $\rho'_{sf}(p) = (J; \rho_{sf})(p) = (J; \rho_{P'H'}; \vartheta)(p) = (\rho'_{P'H'}; \vartheta)(p)$.

QED

Step 2: extensionality

Lemma

If $\lambda 2$ has a set-theoretic model, then there exists an initial T-algebra (P'', H''), i.e. there is a unique homomorphism $\rho''_{sf} : P'' \to s$ for any T-algebra (s, f).

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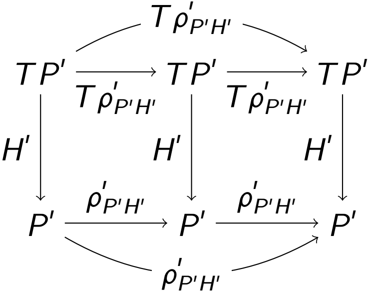
Apply the previous lemma with $s = P'$, $f = H'$, $\vartheta = \rho'_{P'H'}$. Then the following diagram commutes:



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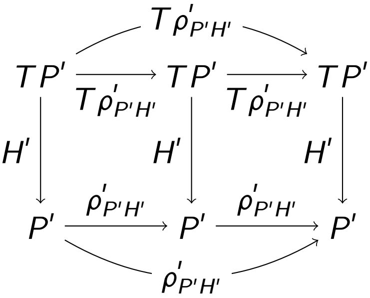

Conclusion: $\rho'_{P'H'} = \rho'_{P'H'} ; \rho'_{P'H'}$.

Step 2: extensionality

- Write $\rho_0' = \rho'_{P'H'}$.
- Define $P'' = \rho_0'[P']$, let $\Gamma : P' \to P''$ be the corestriction of ρ_0', and let $K : P'' \to P'$ be the inclusion map.

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Then:

$$H''; K = TK; H'; \Gamma; K$$

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- We define $H'' : TP'' \to P''$ as $H'' = TK ; H' ; \Gamma$.

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H'' ; K &= TK ; H' ; \Gamma ; K \\
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In summary, the following diagram commutes:

$$\begin{array}{ccccc} TP'' & \xrightarrow{TK} & TP' & \xrightarrow{T\Gamma} & TP'' \\ \downarrow{\scriptstyle H''} & & \downarrow{\scriptstyle H'} & & \downarrow{\scriptstyle H''} \\ P'' & \xrightarrow{K} & P' & \xrightarrow{\Gamma} & P'' \end{array}$$

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Now define $\rho''_{sf} = K; \rho'_{sf}$. Then:

$$H''; \rho''_{sf} = H''; K; \rho'_{sf}$$

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Now define $\rho''_{sf} = K; \rho'_{sf}$. Then:

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\begin{aligned}
H''; \rho''_{sf} &= H''; K; \rho'_{sf} \\
&= TK; H'; \rho'_{sf} \\
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- So $\Gamma; \vartheta'$ is a homomorphism $(P', H') \to (s, f)$, which means that $\rho'_{sf} = \rho'_0; \Gamma; \vartheta'$.

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Thus we have:

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QED

Why extensionality?

Definition (Extensionality)

We say that the parametric polymorphic functions in P' are *extensional* if for all $p, q \in P'$ we have that

$$\forall s \in \mathbf{Set}, \mu_{[\mathbf{p}:\mathbf{P}],\mathbf{W}}[\![\mathbf{p}[\mathbf{s}]]\!][S \mid \mathbf{s} : s][\mathbf{p} : p] = \mu_{[\mathbf{p}:\mathbf{P}],\mathbf{W}}[\![\mathbf{p}[\mathbf{s}]]\!][S \mid \mathbf{s} : s][\mathbf{p} : q] \implies p = q.$$

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- If we assume that the functions in P' are extensional, then $p = q$ whenever $\rho'_{sf}(p) = \rho'_{sf}(q)$ for all s, f.

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- If we assume that the functions in P' are extensional, then $p = q$ whenever $\rho'_{sf}(p) = \rho'_{sf}(q)$ for all s, f.
- But we know that $\rho'_{sf}(p) = \rho'_{sf}(\rho'_0(p))$, so $\rho'_0 = \mathrm{id}_{P'}$.

Why extensionality?

Definition (Extensionality)

We say that the parametric polymorphic functions in P' are *extensional* if for all $p, q \in P'$ we have that

$$\forall s \in \mathbf{Set}, \mu_{[\mathbf{p}:\mathbf{P}],\mathbf{W}}[\![\mathbf{p}[\mathbf{s}]]\!][S \mid \mathbf{s} : s][\mathbf{p} : p] = \mu_{[\mathbf{p}:\mathbf{P}],\mathbf{W}}[\![\mathbf{p}[\mathbf{s}]]\!][S \mid \mathbf{s} : s][\mathbf{p} : q] \implies p = q.$$

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- But we know that $\rho'_{sf}(p) = \rho'_{sf}(\rho'_0(p))$, so $\rho'_0 = \mathrm{id}_{P'}$.
- This means that the last lemma is trivial under this assumption.

Step 3: Lambek's lemma

Theorem (Lambek's lemma)

Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor. If (f, X) is an initial F-algebra, then f is a bijection.

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Proof. By initiality, there is a unique $g : X \to FX$ such that the following diagram commutes:

$$\begin{array}{ccccc}
FX & \xrightarrow{Fg} & F(FX) & \xrightarrow{Ff} & FX \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{g} & FX & \xrightarrow{f} & X
\end{array}$$

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Proof. By initiality, there is a unique $g : X \to FX$ such that the following diagram commutes:

$$\begin{array}{ccccc} FX & \xrightarrow{Fg} & F(FX) & \xrightarrow{Ff} & FX \\ \big\downarrow f & & \big\downarrow Ff & & \big\downarrow f \\ X & \xrightarrow{g} & FX & \xrightarrow{f} & X \end{array}$$

Since id_X is the unique homomorphism $(X, f) \to (X, f)$, we have $g; f = \mathrm{id}_X$.
We also have $f; g = Fg; Ff = F(g; f) = F\,\mathrm{id}_X = \mathrm{id}_{FX}$, so f is a bijection. **QED**

Conclusion

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Proof. Assume there was such a model. Then the functor $T s = (s \to B) \to B$ has an initial algebra (P'', H'').
Lambek's lemma now implies that $H'' : ((P'' \to B) \to B) \to P''$ is a bijection.
This is impossible, since B contains more than 1 element. **QED**

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Since there is no set-theoretic model of $\lambda 2$, we might look for other kinds of models:

- Interpret every type as a set with 0 or 1 elements.
  - This is just a model of second-order propositional logic with proof irrelevance.
- A model based on some other Cartesian closed category.
  - **Example:** The category of complete partial orders and continuous functions.
  - A model has been constructed in [McCracken, 1979].