Problem [B2 from IMO 1972]

f and g are real-valued functions defined on the real line. For all x and y,

$$f(x+y)+f(x-y)=2f(x)g(y).$$

f is not identically zero and $|f(x)| \leq 1$ for all x. Prove that $|g(x)| \leq 1$ for all x.

Solution

Let k be the least upper bound for $|f(x)|$. Suppose $|g(y)| > 1$. Then

$$2k \geq |f(x+y)| + |f(x-y)| \geq |f(x+y)+f(x-y)| = 2|g(y)||f(x)|,$$

so $|f(x)| \leq k/|g(y)|$. In other words, $k/|g(y)|$ is an upper bound for $|f(x)|$ which is less than k. Contradiction.