Problem [B2 from IMO 1972]

f and g are real-valued functions defined on the real line. For all x and y,

$$f(x+y) + f(x-y) = 2f(x)g(y).$$

f is not identically zero and $|f(x)| \leq 1$ for all x. Prove that $|g(x)| \leq 1$ for all x.

Tom Hales' Solution

Note first that $|f(x)||g(y)|^l \leq 1$ for all $l \geq 0$, by induction on l. For the induction step:

$$\begin{aligned} 2|f(x)||g(y)|^{l+1} &= |2f(x)g(y)||g(y)|^l \\ &= |f(x+y) + f(x-y)||g(y)|^l \\ &\leq |f(x+y)||g(y)|^l + |f(x-y)||g(y)|^l \\ &\leq 2 \end{aligned}$$

Now suppose that $|g(y)| > 1$ for some y. We know $f(z) \neq 0$ for some z, but then $|f(z)||g(y)|^l \to \infty$ as $l \to \infty$, contradicting the bound.