Equivalence of Traditional and Nonstandard Definitions of Concepts from Real Analysis

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ACL2(r) is based on Nonstandard Analysis

Rigorous foundations for reasoning about real, complex, infinitesimal, and infinite quantities

- Two versions of the **reals**
  - Standard Reals: ${}^{st}\mathbb{R}$
  - HyperReals: ${}^{\star}\mathbb{R}$

- Standard Reals: ${}^{st}\mathbb{R}$
  - The unique **complete** ordered field.

    . . . .

    Every nonempty subset of ${}^{st}\mathbb{R}$ that is bounded above has a **least upper bound**
  - **No** non-zero infinitesimal elements
  - **No** infinite elements

- HyperReals: ${}^{\star}\mathbb{R}$
  - ${}^{\star}\mathbb{R}$ is a proper field extension of ${}^{st}\mathbb{R}$

    $${}^{st}\mathbb{R} \subsetneqq {}^{\star}\mathbb{R}$$
  - **Has** non-zero infinitesimal elements
  - **Has** infinite elements

- $x \in {}^{\star}\mathbb{R}$ is **infinitesimal**:
  For all positive $r \in {}^{\text{st}}\mathbb{R}$, $(|x| < r)$

  0 is the only infinitesimal in ${}^{\text{st}}\mathbb{R}$

  `(i-small x)` in ACL2(r)

- $x \in {}^{\star}\mathbb{R}$ is **finite**:
  For some $r \in {}^{\text{st}}\mathbb{R}$, $(|x| < r)$

  `(i-limited x)` in ACL2(r)

- $x \in {}^{\star}\mathbb{R}$ is **infinite**:
  For all $r \in {}^{\text{st}}\mathbb{R}$, $(|x| > r)$

  `(i-large x)` in ACL2(r)

- $x, y \in {}^{\star}\mathbb{R}$ are **infinitely close**, $x \approx y$:
  $x - y$ is infinitesimal

  `(i-close x y)` in ACL2(r)

Every (partial) function

$$f : {}^{\mathrm{st}}\mathbb{R}^n \longmapsto {}^{\mathrm{st}}\mathbb{R}^k$$

has an extension

$$^\star f : {}^\star\mathbb{R}^n \longmapsto {}^\star\mathbb{R}^k$$

such that

- For $x_1, \cdots, x_n \in {}^{\mathrm{st}}\mathbb{R}$
  $^\star f(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$

- Every first-order statement about f true in ${}^{\mathrm{st}}\mathbb{R}$ is true about $^\star f$ in $^\star\mathbb{R}$

  Example.
  $(\forall x)[\sin^2(x) + \cos^2(x) = 1]$ is true in ${}^{\mathrm{st}}\mathbb{R}$.

  $(\forall x)[{}^\star\sin^2(x) + {}^\star\cos^2(x) = 1]$ is true in $^\star\mathbb{R}$.

Any (partial) function

$$f : {}^{\text{st}}\mathbb{R}^n \longmapsto {}^{\text{st}}\mathbb{R}^k$$

is said to be **classical**.

- Identify a classical f with its extension ${}^\star f$.

  That is, use f for both the original classical function f and its extension ${}^\star f$.

- Use $(\forall^{\text{st}} x)$ for $(\forall x \in {}^{\text{st}}\mathbb{R})$
  i.e. "for all **standard** x"

  Use $(\exists^{\text{st}} x)$ for $(\exists x \in {}^{\text{st}}\mathbb{R})$
  i.e. "there is some **standard** x"

- $(\forall x)[\sin^2(x) + \cos^2(x) = 1]$ is true in ${}^{\text{st}}\mathbb{R}$
  becomes $(\forall^{\text{st}} x)[\sin^2(x) + \cos^2(x) = 1]$
  (is true in ${}^\star\mathbb{R}$).

  $(\forall x)[{}^\star\sin^2(x) + {}^\star\cos^2(x) = 1]$ is true in ${}^\star\mathbb{R}$
  becomes
  $(\forall x)[\sin^2(x) + \cos^2(x) = 1]$ (is true in ${}^\star\mathbb{R}$).

Real Analysis (i.e., calculus)

Continuous Functions

Function f is **continuous**: Whenever y is "close" to x, then $f(y)$ is "close" to $f(x)$

Three proposed definitions:

Traditional-Standard-1

$$(\forall^{\mathsf{st}} x)(\forall^{\mathsf{st}} \epsilon > 0)(\exists^{\mathsf{st}} \delta > 0)(\forall^{\mathsf{st}} y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Traditional-Hyper-1

$$(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

NonStandard-1

$$(\forall x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

For classical f, ACL2(r) verifies that the two **Traditional** definitions are equivalent.

Traditional-Standard-1

$$(\forall^{\text{st}} x)(\forall^{\text{st}} \epsilon > 0)(\exists^{\text{st}} \delta > 0)(\forall^{\text{st}} y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Traditional-Hyper-1

$$(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Classical Examples

$f_1 : {}^{\star}\mathbb{R} \longmapsto {}^{\star}\mathbb{R}$ defined by $f_1(x) = x$

ACL2(r) verifies that f_1 satisfies all 3 definitions.

Classical Examples

$f_2 : {}^\star\mathbb{R} \longmapsto {}^\star\mathbb{R}$ defined by $f_2(x) = x^2$

ACL2(r) verifies that f_2

- Satisfies both **Traditional** definitions

- Does **not** satisfy **NonStandard-1**

$$(\forall x)(\forall y)(y \approx x \Rightarrow f_2(y) \approx f_2(x))$$

  Let $x = H$ be a positive infinite integer.
  Let $y = H + \frac{1}{H}$
  Then $y - x = \frac{1}{H}$ is an infinitesimal
  but $f_2(y) - f_2(x) = 2 + \frac{1}{H^2}$ is **not** an infinitesimal

Non-Classical Examples

$f_3 : {}^\star\mathbb{R} \longmapsto {}^\star\mathbb{R}$ defined by

$$f_3(x) = \begin{cases} 1 & \text{if } x \in {}^{\text{st}}\mathbb{R} \\ 0 & \text{if } x \notin {}^{\text{st}}\mathbb{R} \end{cases}$$

f_3 is **not** classical:

- $(\forall^{\text{st}} x)(f_3(x) = 1)$ is true, but
  $(\forall x)(f_3(x) = 1)$ is false.

ACL2(r) verifies that f_3

- Satisfies **Traditional-Standard-1**

  $$(\forall^{\text{st}} x)(\forall^{\text{st}} \epsilon > 0)(\exists^{\text{st}} \delta > 0)(\forall^{\text{st}} y)$$
  $$(|y - x| < \delta \Rightarrow |f_3(y) - f_3(x)| < \epsilon)$$

  Both x & y must be standard, so
  $|f_3(y) - f_3(x)| = 0$

$$f_3(x) = \begin{cases} 1 & \text{if } x \in {}^{\text{st}}\mathbb{R} \\ 0 & \text{if } x \notin {}^{\text{st}}\mathbb{R} \end{cases}$$

- Does **not** satisfy **Traditional-Hyper-1**

  $$(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y)$$
  $$(|y - x| < \delta \Rightarrow |f_3(y) - f_3(x)| < \epsilon)$$

  Let $x = 0$ and $\epsilon = 1$
  Let h be a positive infinitesimal
  For any $\delta > 0$, let $y = \min\left(\frac{\delta}{2}, h\right)$
  Then $y \leq \frac{\delta}{2} < \delta$ & y is an infinitesmal
  So $|f_3(y) - f_3(x)| = 1$

- Does **not** satisfy **NonStandard-1**

  $$(\forall x)(\forall y)(y \approx x \Rightarrow f_3(y) \approx f_3(x))$$

  Let $x = 0$
  Let $y = h$ be an infinitesimal
  Then $y - x = h$ is an infinitesimal
  but $f_3(y) - f_3(x) = -1$ is **not** an infinitesimal

Non-Classical Examples

$f_4 : {}^\star\mathbb{R} \longmapsto {}^\star\mathbb{R}$ defined by

$$f_4(x) = \begin{cases} 0 & \text{if } x \approx 0 \\ 1 & \text{if } x \not\approx 0 \wedge x < 1 \\ 1 + h & \text{if } x \geq 1 \end{cases}$$

- h is a positive infinitesimal

- f_4 is **not** classical

- f_4 does **not** satisfy
  Traditional-Standard-1
  Let $x = 0$ and $\epsilon = 1$.
  For any standard $\delta > 0$, let $y = \frac{\delta}{2}$.
  Then y is standard, $|y - x| = \frac{\delta}{2} < \delta$,
  $f_4(x) = 0$, $f_4(y) = 1$ or $1 + h$, and
  $|f_4(y) - f_4(x)| \geq 1$.

$$f_4(x) = \begin{cases} 0 & \text{if } x \approx 0 \\ 1 & \text{if } x \not\approx 0 \wedge x < 1 \\ 1 + h & \text{if } x \geq 1 \end{cases}$$

- f_4 does **not** satisfy **Traditional-Hyper-1**
  Let $x = 1$ and $\epsilon = h$.
  For any $\delta > 0$, let $y = 1 - \frac{\delta}{2}$. Then
  $|y - x| = \frac{\delta}{2} < \delta$, $f_4(y) = 1$ or 0,
  $f_4(x) = 1 + h$, and $|f_4(y) - f_4(x)| \geq h$.

- f_4 satisfies **NonStandard-1**
  If $y \approx 0 \approx x$, then $f_4(y) = 0 = f_4(x)$.
  If $y \approx x$ but x is not an infinitesimal, then
  $f_4(y) = 1$ or $1 + h$ and $f_4(x) = 1$ or $1 + h$,
  so $f_4(x) \approx f_4(y)$

Recall $f_2(x) = x^2$ does not satisfy
NonStandard-1

$$(\forall x)(\forall y)(y \approx x \Rightarrow f_2(y) \approx f_2(x))$$

Let $x = H$ be a positive **infinite** integer.
Let $y = H + \frac{1}{H}$
Then $y - x = \frac{1}{H}$ is an infinitesimal
but $f_2(y) - f_2(x) = 2 + \frac{1}{H^2}$ is **not** an infinitesimal

Modify **NonStandard-1** so that $f_2(x) = x^2$ satisfies the new definition.

Require x to be **standard real**

NonStandard-2

$$(\forall^{\mathrm{st}} x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

Continuous Functions

For **classical** f, ACL2(r) verifies these three definitions are equivalent.

Traditional-Standard-1

$$(\forall^{\mathsf{st}} x)(\forall^{\mathsf{st}} \epsilon > 0)(\exists^{\mathsf{st}} \delta > 0)(\forall^{\mathsf{st}} y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Traditional-Hyper-1

$$(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

NonStandard-2

$$(\forall^{\mathsf{st}} x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

Modify **Traditional-Standard-1** (and **Traditional-Hyper-1**) into something equivalent to **NonStandard-1**

Rearrange the quantifiers

Traditional-Standard-1

$$(\forall^{\text{st}} x)(\forall^{\text{st}} \epsilon > 0)(\exists^{\text{st}} \delta > 0)(\forall^{\text{st}} y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Traditional-Standard-2

$$(\forall^{\text{st}} \epsilon > 0)(\exists^{\text{st}} \delta > 0)(\forall^{\text{st}} x)(\forall^{\text{st}} y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Uniformly Continuous Functions

For **classical** f, ACL2(r) verifies these three definitions are equivalent.

Traditional-Standard-2

$$(\forall^{\mathsf{st}} \epsilon > 0)(\exists^{\mathsf{st}} \delta > 0)(\forall^{\mathsf{st}} x)(\forall^{\mathsf{st}} y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Traditional-Hyper-2

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

NonStandard-1

$$(\forall x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

For **classical** f,
the **Traditional-hyper** definitions are "purely" **classical**.

Only classical functions are mentioned in these definitions: f, $<$, $>$, $-$, $|\ |$

Traditional-Hyper-1

$$(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

Traditional-Hyper-2

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

For **classical** f,
the **NonStandard** definitions are **not** "purely" **classical**

Non-classical functions are mentioned in these definitions: $\approx$, $\forall^{\mathsf{st}}$

NonStandard-1

$$(\forall x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

NonStandard-2

$$(\forall^{\mathsf{st}} x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

For **classical** f,
the **NonStandard** definitions are "simpler" than **Traditional-hyper** definitions.

Look at the alternations of the quantifiers.

Traditional-Hyper-2

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(\forall y)$$
$$(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

NonStandard-1

$$(\forall x)(\forall y)(y \approx x \Rightarrow f(y) \approx f(x))$$

Limits of Functions

$$\lim_{x \to a} f(x) = L$$

For **standard** numbers, a and L, and **classical** function f, ACL2(r) verifies these three definitions are equivalent.

Traditional-Standard-3

$$(\forall^{\text{st}} \epsilon > 0)(\exists^{\text{st}} \delta > 0)(\forall^{\text{st}} x)$$
$$(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon)$$

Traditional-Hyper-3

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)$$
$$(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon)$$

NonStandard-3

$$(\forall x)((x \approx a \wedge x \neq a) \Rightarrow f(x) \approx L)$$

The derivative of f is f'

For **classical** f and f', ACL2(r) verifies these two definitions are equivalent.

Traditional-Hyper-4

$$
\begin{array}{l}
(\forall x)(\forall \epsilon > 0)(\exists \delta > 0)(\forall y) \\
\quad \left[\begin{array}{l} 0 < |y| < \delta \Rightarrow \\ \qquad \left| \frac{f(x+y)-f(x)}{y} - f'(x) \right| < \epsilon \end{array} \right]
\end{array}
$$

NonStandard-4

$$
\begin{array}{l}
(\forall^{\text{st}} x)(\forall x_1) \\
\quad \left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{f(x_1)-f(x)}{x_1-x} \approx f'(x) \end{array} \right]
\end{array}
$$

Nelson's Theorem 5.6 from his paper "Internal Set Theory: A New Approach to Nonstandard Analysis"

Theorem. Let $f : I \mapsto {}^\star\mathbb{R}$ where I is an interval. If f is differentiable on I, then f' is continuous on I.

Here Nelson means for f to be classical

What about?

$$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f' is **not** continuous at $x = 0$

The derivative of f is f'

Nelson's definition **differs** from **NonStandard-4**.
They are **not** equivalent.

Nelson

$$(\forall^{\text{st}} x)(\forall x_1)(\forall x_2)$$
$$\left[\begin{array}{c}(x_1 \approx x \wedge x_2 \approx x \wedge x_1 \neq x_2) \Rightarrow \\ \frac{f(x_2)-f(x_1)}{x_2-x_1} \approx f'(x)\end{array}\right]$$

NonStandard-4

$$(\forall^{\text{st}} x)(\forall x_1)$$
$$\left[\begin{array}{c}(x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \frac{f(x_1)-f(x)}{x_1-x} \approx f'(x)\end{array}\right]$$

Using Nelson's definition, ACL2(r) verifies Theorem 5.6

ACL2 Functional-Instances allow free-varables in lambda expressions.

```
(encapsulate
(((c) => *))

(local (defun
         c ()
         2))

(defthm
  int-c->1
  (and (integerp (c))
       (> (c) 1))))

(defthm
 mod-c-thm-1
 (implies (and (integerp x)
               (not (equal (mod x (c)) 0)))
          (equal (mod (- x) (c))
                 (- (c) (mod x (c))))))
```

Replace constant (c) with variable n

```
(thm
(implies (and (integerp x)
              (integerp n)
              (> n 1)
              (not (equal (mod x n) 0)))
         (equal (mod (- x) n)
                (- n (mod x n))))
:hints
 (("Goal"
   :by (:functional-instance
        mod-c-thm-1
        (c (lambda () (if (and (integerp n)
                               (> n 1))
                          n
                          (c))))))))
;; Free variable n allowed
;;      in functional instance
```

Q.E.D.

Some ACL2(r) Functional-Instances do not allow free-varables in lambda expressions.

```
(defstub  ;;classical f
 f (x) t)

(defthm-std
  standardp-f
  (implies (standardp x)
           (standardp (f x))))

(thm  ;;not a theorem
(implies (standardp x)
         (standardp (+ x y)))
:hints (("Goal"
         :by (:functional-instance
              standardp-f
              (f (lambda (x)(+ x y)))))))

(thm  ;; a theorem
(let ((x 0)
      (y (i-large-integer)))
  (not (implies (standardp x)
                (standardp (+ x y))))))
```

```
(thm  ;;not a theorem
(implies (standardp x)
         (standardp (+ x y)))
:hints (("Goal"
         :by (:functional-instance
              standardp-f
              (f (lambda (x)(+ x y)))))))
```

```
ACL2 Error in ( THM ...):  Your functional
substitution contains one or more free
occurrences of the variable Y in its range.

Alas, the formula you wish to functionally
instantiate is not a classical formula,
(IMPLIES (STANDARDP X) (STANDARDP (F X))).

Free variables in lambda expressions are
only allowed when the formula to be
instantiated is classical, since these
variables may admit non-standard values,
for which the theorem may be false.
```

```
(encapsulate
((fn1 (x) t)         ;;classical
 (fn1' (x) t)        ;;classical
 (I1 () t)           ;;classical
 (c1 () t))          ;;classical
  . . .
 (defun-sk           ;;non-classical
   NonStd-deriv-fn1=fn1' ()
```

$$(\forall^{\mathsf{st}} x \in \mathtt{I1})(\forall x_1 \in \mathtt{I1}) \left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{\mathtt{fn1}(x_1) - \mathtt{fn1}(x)}{x_1 - x} \approx \mathtt{fn1'}(x) \end{array} \right]$$

```
 ) ;;end defun-sk

 (defthm          ;;assume this holds
   Thm-NonStd-deriv-fn1=fn1'
   (NonStd-deriv-fn1=fn1'))
 ) ;; end encapsulate
```

Continue after the encapsulate

```
(defun      ;;classical
  c1*fn1 (x)
  (* (c1)(fn1 x)))

(defun      ;;classical
  c1*fn1' (x)
  (* (c1)(fn1' x)))
. . .
 (defun-sk    ;;non-classical
   NonStd-deriv-c1*fn1=c1*fn1' ()
```

$$(\forall^{\text{st}} x \in \texttt{I1})(\forall x_1 \in \texttt{I1})$$

$$\left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{\texttt{c1*fn1}(x_1) - \texttt{c1*fn1}(x)}{x_1 - x} \approx \texttt{c1*fn1'}(x) \end{array} \right]$$

```
 ) ;;end defun-sk

(defthm
  Thm-NonStd-deriv-c1*fn1=c1*fn1'
  (NonStd-deriv-c1*fn1=c1*fn1'))
```

(acl2-ln x) is the natural logarithm of $x > 0$.

The derivative of (acl2-ln x) is $\frac{1}{x}$:

```
(defun-sk          ;;non-classical
   NonStd-deriv-ln=1/x ()
```

$$(\forall^{st} x > 0)(\forall x_1 > 0) \left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{\texttt{acl2-ln}(x_1) - \texttt{acl2-ln}(x)}{x_1 - x} \approx \frac{1}{x} \end{array} \right]$$

```
) ;;end defun-sk

(defthm
  Thm-NonStd-deriv-ln=1/x'
  (NonStd-deriv-ln=1/x))
```

(log2 x) is the logarithm, base 2, of $x > 0$.

Use acl2-ln to define log2

```
(defun
 log2 (x)
 (/ (acl2-ln x)(acl2-ln 2)))
```

The derivative of (log2 x) is $\frac{1}{(\text{acl2-ln } 2)\cdot x}$:

```
(defun-sk          ;;non-classical
   NonStd-deriv-log2 ()
```

$$(\forall^{\text{st}} x > 0)(\forall x_1 > 0) \left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{\text{log2}(x_1) - \text{log2}(x)}{x_1 - x} \approx \frac{1}{(\text{acl2-ln } 2)\cdot x} \end{array} \right]$$

```
) ;;end defun-sk

;;Use Functional-Instance to prove
 (defthm
   Thm-NonStd-deriv-log2
   (NonStd-deriv-log2))
```

```
;;Use Functional-Instance to prove
 (defthm
   Thm-NonStd-deriv-log2
   (NonStd-deriv-log2)
   :hints
    (("Goal"
       :by
        (:functional-instance
         Thm-NonStd-deriv-c1*fn1=c1*fn1'
         (fn1 acl2-ln)
         (fn1' (lambda (x)(/ x)))
         (I1 (Interval 0 +infinity))
         (c1 (lambda ()(/ (acl2-ln 2))))
         (c1*fn1 log2)
         (c1*fn1' (lambda (x)
                          (/ (* (acl2-ln 2)
                                x))))
         . . .
        ))))
```

The lambda expressions have no free variables

The functional substitution

```
(fn1 acl2-ln)
(fn1' (lambda (x)(/ x)))
(I1 (Interval 0 +infinity))
(c1 (lambda ()(/ (acl2-ln 2))))
(c1*fn1 log2)
(c1*fn1' (lambda (x)
                  (/ (* (acl2-ln 2)
                        x))))
```

The functional-instance of this statement

$$(\forall^{\text{st}} x \in \texttt{I1})(\forall x_1 \in \texttt{I1}) \left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{\texttt{c1*fn1}(x_1) - \texttt{c1*fn1}(x)}{x_1 - x} \approx \texttt{c1*fn1'}(x) \end{array} \right]$$

is this statement

$$(\forall^{\text{st}} x > 0)(\forall x_1 > 0) \left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x) \Rightarrow \\ \qquad \frac{\texttt{log2}(x_1) - \texttt{log2}(x)}{x_1 - x} \approx \frac{1}{(\text{acl2-ln } 2) \cdot x} \end{array} \right]$$

(log b x) is the logarithm, base b > 1, of x > 0.

Use acl2-ln to define log

```
(defun
 log (b x)
 (/ (acl2-ln x)(acl2-ln b)))
```

The derivative of (log b x) is $\frac{1}{\text{(acl2-ln b)}\cdot\text{x}}$:

```
 (defun-sk          ;;non-classical
    NonStd-deriv-log (b)
```

$$(\forall^{\text{st}} x > 0)(\forall x_1 > 0)$$
$$\left[\begin{array}{l} (x_1 \approx x \wedge x_1 \neq x \wedge b > 1) \Rightarrow \\ \qquad \frac{\texttt{log}(b\ x_1) - \texttt{log}(b\ x)}{x_1 - x} \approx \frac{1}{\text{(acl2-ln b)}\cdot x} \end{array} \right]$$

```
 ) ;;end defun-sk

;;Use Functional-Instance to prove
 (defthm
   Thm-NonStd-deriv-log
   (NonStd-deriv-log b))
```

```
;;Use Functional-Instance to prove
 (defthm
   Thm-NonStd-deriv-log
   (NonStd-deriv-log)
   :hints
    (("Goal"
       :by
        (:functional-instance
         Thm-NonStd-deriv-c1*fn1=c1*fn1'
         . . .
         (c1 (lambda ()
                (if (and (realp b) (< 1 b))
                    (/ (acl2-ln b))
                    (/ (acl2-ln 2)))))
         (c1*fn1 (lambda (x)
                    (if (and (realp b) (< 1 b))
                        (log b x)
                        (log 2 x))))
         . . .
        ))))
```

The lambda expressions in the previous slide have free variable b, so ACL2(r) will not permit this `functional-instance`, because `NonStd-deriv-log` is not classical.

However, ACL2(r) will permit the equivalent result using the traditional hyper definition (**Traditional-Hyper-4**), because that does not use any non-classical terms (e.g., $\approx$).