Raising (Ir)rationals to (Ir)rational Exponents

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ACL2(r) is based on Nonstandard Analysis

Rigorous foundations for reasoning about real, complex, infinitesimal, and infinite quantities

Recall

- **rationals** are fractions

- **irrationals** are reals that are **not** fractions

ACL2(r) can verify

- These are irrational:
  $\sqrt{3}$, $\sqrt{2}$, $-\sqrt{2}$, $\frac{1}{\sqrt{2}}$, $2 \cdot \sqrt{2}$.

- There are irrationals a and b so that

  - $a + b$ is rational
    $-\sqrt{2} + \sqrt{2} = 0$

  - $a \cdot b$ is rational
    $\frac{1}{\sqrt{2}} \cdot \sqrt{2} = 1$

Question.

- Are there irrationals a and b so that a^b is rational?

What about $\sqrt{2}^{\sqrt{2}}$?

Is $\sqrt{2}^{\sqrt{2}}$ rational or is $\sqrt{2}^{\sqrt{2}}$ irrational?

ACL2(r) can verify this Folklore:

Either (1) $\sqrt{2}^{\sqrt{2}}$ is rational
OR (2) $\sqrt{2}^{\sqrt{2}}$ is irrational.

1. $\sqrt{2}^{\sqrt{2}}$ is rational
   Let $a = \sqrt{2} = b$.
   Then $a^b = \sqrt{2}^{\sqrt{2}}$ is rational

2. $\sqrt{2}^{\sqrt{2}}$ is irrational
   Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.
   Then $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$ is rational

In both cases, there are irrationals a & b so that a^b is rational.

This Folklore argument does not answer this question:

Is $\sqrt{2}^{\sqrt{2}}$ rational or is $\sqrt{2}^{\sqrt{2}}$ irrational?

Hilbert's Seventh Problem (1900) asks for a proof of

The expression α^{β}, for an algebraic base α and an irrational algebraic exponent β, e.g., the number $2^{\sqrt{2}}$ or $e^{\pi} = i^{-2i}$, always represents a transcendental or at least an irrational number.

- A complex number, x, is **algebraic** if it satisfies some equation of the form $a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0 = 0$ with **integer** coefficients

- A complex number is **transcendental** if it is **not** algebraic.

The Gelfond–Schneider Theorem (1934–35) provides the answer:

If α and β are algebraic with $\alpha \neq 0$, $\alpha \neq 1$, and if β is not a real rational number, then α^β is transcendental.

All of these are transcendental (and also irrational): $\sqrt{2}^{\sqrt{2}}$, $2^{\sqrt{2}}$, 2^i, $e^\pi = i^{-2i}$

Future Work: Use ACL2(r) to verify Gelfond–Schneider Theorem.

We have not yet proved the Gelfond–Schneider Theorem

What did we do?

Suppose x and y are either rational or irrational reals.

The result of x^y may be either rational or irrational.

We verify this in ACL2(r) by giving explicit witnesses to all the possibilities.

Exponentiation in ACL2(r)

- `(expt r n)` raises a **complex** base to an **integer** exponent

  $(\texttt{expt}\ r\ n) = (\texttt{fix}\ r)^{(\texttt{ifix}\ n)}$

- `(acl2-exp x)` raises e to a **complex** exponent

  Defined in ACL2(r) via power series:

$$\begin{aligned}(\texttt{acl2}-\texttt{exp}\ x) &= e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\end{aligned}$$

- `(acl2-ln x)` is a natural logarithm of a **complex** $\texttt{x} \neq 0$.

  - $e^x : [0, \infty) \longmapsto [1, \infty)$ is **1-1** and **continuous**.

    ACL2(r) allows the inverse function to be defined:

    $\ln^{\geq 1} : [1, \infty) \longmapsto [0, \infty)$

  - Extend the domain of $\ln^{\geq 1}$ to $(0, \infty)$:

$$\ln^{+}(x) = \begin{cases} \ln^{\geq 1}(x) & \text{if } x \in [1, \infty) \\ -\ln^{\geq 1}(1/x) & \text{if } x \in (0, 1) \end{cases}$$

  - Extend the domain to all non-zero complex numbers $a + b \cdot i = re^{i\theta}$

    $(\texttt{acl2-ln}\ re^{i\theta}) = \ln^{+}(r) + i\theta$

- `(raise x y)` raises a **complex** base
  to a **complex** exponent

  - Definition in ACL2(r) for
    $(\texttt{raise}\ x\ y) = x^y$:

$$x^y = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } x = 0 \ \&\ y \neq 0 \\ e^{y \ln(x)} & \text{if } x \neq 0 \ \&\ y \neq 0 \end{cases}$$

  - For integer n, $(\texttt{raise}\ x\ n) = (\texttt{expt}\ x\ n)$

  - For real $x \geq 0$,
    $(\texttt{raise}\ x\ 1/2) = x^{1/2} = \sqrt{x}$

  - $(\texttt{raise}\ -1\ 1/2) = (-1)^{1/2} = i$

- `(acl2-log b x)` is $\log_{\mathtt{b}}(\mathtt{x})$ for **real** $\mathtt{x} > 0$
  and **real** $\mathtt{b} > 0$ and $\mathtt{b} \neq 1$.

  ACL2(r) definition based on

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

- $\log_2(3)$ is irrational
  - Suppose $\log_2(3) = p/q$
  - Then $2^{p/q} = 2^{\log_2(3)} = 3$
  - Then $2^p = 3^q$
  - If $p = 0$, then $1 = 2^p = 3^q \geq 3$
  - If $p < 0$, then $1 > 2^p = 3^q \geq 3$
  - If $p > 0$, then **even** $= 2^p = 3^q =$ **odd**

- $2 \cdot \log_2(3)$ is also irrational

- $\frac{1}{2} \cdot \log_2(3)$ is also irrational

rational base to **rational** exponent

- $2^2 = 4$
- $2^{1/2} = \sqrt{2}$

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rational base to **irrational** exponent

- Exclude base = 0,1
- $2^{\log_2(3)} = 3$
- $2^{1/2 \cdot \log_2(3)} = \sqrt{3}$

irrational base to **rational** exponent

- Exclude exponent = 0,1
- $\left(\sqrt{2}\right)^2 = 2$
- $\left(\sqrt{2}\right)^3 = 2 \cdot \sqrt{2}$

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irrational base to **irrational** exponent

- $\left(\sqrt{2}\right)^{2 \cdot \log_2(3)} = 3$
- $\left(\sqrt{2}\right)^{\log_2(3)} = \sqrt{3}$