Böhm's Theorem, Church's Delta, Numeral Systems, and Ershov Morphisms

Richard Statman & Henk Barendregt

Department of Mathematics, Carnegie-Mellon University, Pittsburgh PA, USA
Faculty of Science, Radboud University, Nijmegen, The Netherlands

Abstract. In this note we work with untyped lambda terms under β-conversion and consider the possibility of extending Böhm's theorem to infinite RE (recursively enumerable) sets. Böhm's theorem fails in general for such sets $\mathcal{V}$ even if it holds for all finite subsets of it. It turns out that generalizing Böhm's theorem to infinite sets involves three other superficially unrelated notions; namely, Church's delta, numeral systems, and Ershov morphisms. Our principal result is that Böhm's theorem holds for an infinite RE set $\mathcal{V}$ closed under beta conversion iff $\mathcal{V}$ can be endowed with the structure of a numeral system with predecessor iff there is a Church delta (conditional) for $\mathcal{V}$ iff every Ershov morphism with domain $\mathcal{V}$ can be represented by a lambda term.

1. Introduction

We suppose the reader knows some lambda calculus, as e.g. in Barendregt [1984], Chapters 6, 7, 8 and 10.

1. Definition. (i) The set of untyped closed lambda terms is denoted by $\Lambda^{\varnothing}$. A *combinator* is an element of $\Lambda^{\varnothing}$.

(ii) We denote congruence under beta conversion by $=$.

(iii) We write $:=$ for "equal by definition".

(iv) We define the following combinators.

$$\mathbf{c}_n := \lambda fx.f^n x, \qquad \text{the } \textit{Church numerals}.$$
$$\mathsf{U}^n_k := \lambda x_1 \dots x_n.x_k, \qquad \text{for } 1 \le k \le n, \text{ the } \textit{projections}.$$
$$\Omega := (\lambda x.xx)(\lambda x.xx).$$

(v) For lambda terms $\vec{P} = P_1, \dots, P_n$ we write

$$\langle P_1, \dots, P_n \rangle := \lambda z.zP_1 \dots P_n.$$

Note that

$$\langle P_1, \dots, P_n \rangle = \langle Q_1, \dots, Q_n \rangle \;\Leftrightarrow\; P_1 = Q_1 \;\&\; \dots \;\&\; P_n = Q_n.$$

The classical theorem of Böhm states the following.

2. Theorem (Böhm [1968]). For all combinators M_1 and M_2 having a β-nf (normal form) the following are equivalent.

(i) For all combinators N_1, N_2 there exist combinators $\vec{P}$ such that

$$M_1\vec{P} = N_1 \;\&\; M_2\vec{P} = N_2.$$

(ii) There exists a combinator F such that

$$FM_1 = \lambda xy.x \ \& \ FM_2 = \lambda xy.y.$$

(iii) $M_1 = M_2$ is inconsistent with $\lambda\beta$.
(iv) $M_1 = M_2$ is inconsistent with $\lambda\beta\eta$.
(v) M_1 and M_2 have distinct $\beta\eta$-nfs (normal forms).

PROOF. (i)⇒(ii) Let $N_i := \lambda x_1 x_2.x_i$, for $1 \le i \le 2$. By (i) there are $\vec{P}$ such that $M_i\vec{P} = N_i$. Take $F := \lambda m.m\vec{P}$.

(ii)⇒(iii) From the equation $M_1 = M_2$ one can by (ii) derive $\lambda xy.x = \lambda xy.y$, from which one can derive any equation; all derivations using just $\lambda\beta$.

(iii)⇒(iv) Trivial.

(iv)⇒(v) By the hypothesis that M_1, M_2 have β-nf and Barendregt [1984], Corollary 15.1.5, it follows that M_1, M_2 have $\beta\eta$-nfs. If these were equal, then $M_1 =_{\beta\eta} M_2$ and hence $M_1 = M_2$ would be consistent.

(v)⇒(i) This is the core of Böhm's theorem. A proof can be found in Barendregt [1984], Theorem 10.4.2. ■

The equivalences do not hold for arbitrary terms M_1, M_2, not in β-nf.

3. REMARK. Referring to Theorem 2 one has the following.

1. In the list of equivalences on could add (iv$^{\mathrm{a}}$) $M \ne_{\beta\eta} N$. Indeed, (iv) ⇒ (iv$^{\mathrm{a}}$) ⇒ (v).
2. The implications (i)⇒(ii), (ii)⇒(iii), (iii)⇒(iv) and (v)⇒(iv) hold trivially for all M_1, M_2. Also (v)⇒(i) holds (but not trivially), as the condition of normalizability holds by assumption.
3. In general (iv)⇏(v). One has $\Omega = \mathsf{I}$ is consistent with $\lambda\beta\eta$, as follows by the technique of Jacopini [1975], but Ω does not have a $\beta\eta$-nf.
4. Similarly (iv)⇏(iii). For example the set of equations

$$\{\Omega\mathsf{I} = \mathsf{U}^2_1, \Omega\mathbf{c}_1 = \mathsf{U}^2_2\}$$

is consistent with $\lambda\beta$, see Barendregt [1984], Corollaries 15.3.6 and 15.3.7. But the set is inconsistent with $\lambda\beta\eta$, as $\mathsf{I} =_{\beta\eta} \mathbf{c}_1$. Hence $\langle\Omega\mathsf{I}, \Omega\mathbf{c}_1\rangle = \langle\mathsf{U}^2_1, \mathsf{U}^2_2\rangle$ is consistent with $\lambda\beta$, but not with $\lambda\beta\eta$.
5. As to (iii)⇏(ii), the equation $\Omega_3 = \mathsf{I}$, with $\Omega_3 \equiv (\lambda x.xxx)(\lambda x.xxx)$, is inconsistent as shown in Jacopini [1975]. But if $F\Omega_3 = \lambda xy.x$ and $F\mathsf{I} = \lambda xy.y$ for some F, then by Barendregt [1984], Proposition 14.3.24, it follows that either Ω_3 is solvable, which it isn't, or $\forall M.FM = \lambda xy.x$, which contradicts the second equation.
6. (i)⇏(v) Let $M_1 = \langle\lambda xy.x, \Omega\rangle, M_2 = \langle\lambda xy.y, \Omega\rangle$. Taking $\vec{P} := \mathsf{U}^2_1, N_1, N_2$ one has $M_1\vec{P} = N_1 \ \& \ M_2 P = N_2$, but M_i has no β-nf.
7. We believe that (ii)⇒(i) also holds in general.

Let $\mathcal{F} = \{M_1, \ldots, M_n\}$ be a finite set of combinators.

4. DEFINITION. $\mathcal{F}$ is called *separable* iff there exists a combinator F such that

$$FM_1 = \mathsf{U}^n_1 \ \& \ \ldots \ \& \ FM_n = \mathsf{U}^n_n.$$

This notion of separability has a different but equivalent definition.

5. LEMMA. Let $\mathcal{F} = \{M_1, \dots, M_n\}$ be a finite set of arbitrary combinators. Then the following are equivalent.

(i) There exists an $F \in \Lambda^{\emptyset}$ such that

$$FM_1 = \mathsf{U}_1^n \;\&\; \dots \;\&\; FM_n = \mathsf{U}_n^n.$$

(ii) There exists an $F \in \Lambda^{\emptyset}$ such that

$$FM_1 = \mathbf{c}_1 \;\&\; \dots \;\&\; FM_n = \mathbf{c}_n.$$

PROOF. (i)⇒(ii) Given F, take $F' := \lambda m.Fm\mathbf{c}_1 \dots \mathbf{c}_n$.

(ii)⇒(i) Assume (ii) for some F. By induction on n one can show that for some G_n one has

$$1 \le k \le n \;\Rightarrow\; G_n\mathbf{c}_k = \mathsf{U}_k^n \qquad (*)$$

For $n = 0$ there is nothing to prove and for $n = 1$ we can take $G_1 = \mathsf{K}\mathsf{U}_1^1$. Suppose that G_n has been defined and satisfies $(*)$. Define

$$\begin{aligned} G_{n+1} &:= \lambda c.\,\texttt{If}\;\;(\texttt{Zero?}\;c)\;\;\texttt{then}\;\;\mathsf{U}_1^{n+1}\texttt{else}\;\;(G_n(P^-c)\mathsf{U}_2^{n+1}\dots\mathsf{U}_{n+1}^{n+1}) \\ &:= \lambda c.(\texttt{Zero?}\;c)\mathsf{U}_1^{n+1}(G_n(P^-c)\mathsf{U}_2^{n+1}\dots\mathsf{U}_{n+1}^{n+1}), \end{aligned}$$

where $(\texttt{If}\;\;B\;\;\texttt{then}\;\;P\;\;\texttt{else}\;\;Q) \equiv BPQ$, $\texttt{Zero?} \equiv \lambda n.n(\lambda x.\mathsf{U}_2^2)\mathsf{U}_1^2$ so that

$$\texttt{Zero?}\mathbf{c}_0 = \mathsf{U}_1^2,\;\; \texttt{Zero?}\mathbf{c}_{k+1} = \mathsf{U}_2^2$$

and $P^- \in \Lambda^{\emptyset}$ is a representation of the predecessor function for the Church numerals. Then G_{n+1} works. Now we can take $F' := \lambda m.G_n(Fm)$. ■

For infinite sets the two ways of defining the notions of separability are no longer equivalent. Separability for possibly infinite sets has to be defined as the existence of a definable 1-1 map (modulo β-conversion) to the Church numerals.

6. DEFINITION. $\mathcal{V}$ is said to be *separable* if for some combinator D one has

(i) $\forall M \in \mathcal{V} \exists n \in \mathbb{N}.DM = \mathbf{c}_n$.

(ii) $\forall M, N \in \mathcal{V}.[DM = DN \;\Leftrightarrow\; M = N]$.

For such $\mathcal{F}$ one has the following generalizaton of Theorem 2.

7. THEOREM (Böhm, Dezani-Ciancaglini, Peretti and Ronchi [1979]). For all $\mathcal{F} = M_1, \dots, M_n$, where each M_i has a β-nf, the following are equivalent.

(i) For all combinators $N_1, \dots, N_n$ there exist combinators $\vec{P}$ such that

$$M_1\vec{P} = N_1 \;\&\; \dots \;\&\; M_n\vec{P} = N_n.$$

(ii) $\mathcal{F}$ is separable.

(iii) $M_p = M_q$ is inconsistent with $\lambda\beta$, for $1 \le p, q \le n$ with $p \ne q$.

(iv) $M_p = M_q$ is inconsistent with $\lambda\beta\eta$, for $1 \le p, q \le n$ with $p \ne q$.

(v) The $M_1, \dots, M_n$ have pairwise distinct $\beta\eta$-nfs.

PROOF. Again, the only non-trivial implication is (v)⇒(i) and is proved in Böhm, Dezani-Ciancaglini, Peretti and Ronchi [1979], see Barendregt [1984], Corollary 10.4.14. ■

In trying to generalize Theorem 7 by dropping the requirement that $\mathcal{F}$ is finite or that its elements have a nf, several problems arise.

8. REMARK. (i) For finite sets $\mathcal{F}$ of combinators having a β-nf the property of consisting of pairwise $\beta\eta$-inconvertible terms is equivalent to separability. Indeed, the Church-Rosser Theorem implies that

$$M \neq_{\beta\eta} N \;\Leftrightarrow\; M, N \text{ have distinct } \beta\eta\text{-nfs},$$

hence Theorem 7 applies. If we drop the requirement that the elements of $\mathcal{F}$ have a β-nf, then this is no longer true. For example this is the case with

$$\mathcal{F} = \{\Omega\mathbf{c}_1, \ldots, \Omega\mathbf{c}_n\}.$$

This set consists of pairwise β-inconvertible terms, but is not separable, as follows from the Genericity Lemma in Barendregt [1984], Proposition 14.3.24.

(ii) For infinite sets $\mathcal{F}$ of terms having pairwise distinct $\beta\eta$-nf separability does not necessarily hold either. An example is the collection of projections

$$\mathcal{F} = \{\mathsf{U}^n_k \mid n \in \mathbb{N} \;\&\; 1 \leq k \leq n\}.$$

The set clearly consists of pairwise distinct $\beta\eta$-nfs. This $\mathcal{F}$ is such that each finite subset of it is separable but not the whole set itself. That $\mathcal{F}$ is not separable can be seen as follows. Suppose that F is a combinator which maps the set of projections into the Church numerals injectively. Let U range over the projections. FU is $\beta\eta$-convertible to a Chruch numeral which is a λI-term for all U, except possibly one (that is mapped to $\mathbf{c}_1$). Then for those U it follows, by η-postponement, see Barendregt [1984], Corollary 15.1.6, and the obvious fact that η-conversion does not change the status of being a λI-term, that $FU =_\beta N$ for some λI-term N in nf. Consider a standard reduction $FU \twoheadrightarrow_\beta N$. Then $Fx \twoheadrightarrow_\beta N'$, with $x \in \mathrm{FV}(N')$ (as F is not a constant map) and to the left of the leftmost occurrence of x in N' there is no expression of the form $(\lambda y.P)$. But then for almost all U one has that $FU = N'[x := U]$ is not convertible to a λI-term, a contradiction.

A characterization of separability for finite $\mathcal{F}$, possibly containing terms without normal form is due to Coppo, Dezani-Ciancaglini and Ronchi [1978] and can be found also in Barendregt [1984], Theorem 10.4.13. To give a flavor of that theorem we give some of its consequences, rather than repeating its precise formulation.

1. The set $\left\{\begin{array}{l}\lambda x.x\mathbf{c}_0\Omega,\\ \lambda x.x\mathbf{c}_1\Omega\end{array}\right\}$ is separable.
2. $\left\{\begin{array}{l}\lambda x.x(\lambda y.y\Omega),\\ \lambda x.x(\lambda y.y\mathbf{c}_0)\end{array}\right\}$ is not separable.

3. $\left\{\begin{array}{l}\lambda x.x(\lambda y.y\mathbf{c}_0\Omega(\lambda z.z\Omega),\\ \lambda x.x(\lambda y.y\mathbf{c}_1\Omega(\lambda z.z\mathbf{c}_1),\\ \lambda x.x(\lambda y.y\mathbf{c}_1\Omega(\lambda z.z\mathbf{c}_2)\end{array}\right\}$ is separable.

4. $\left\{\begin{array}{l}\lambda x.x\mathbf{c}_0\mathbf{c}_0\Omega,\\ \lambda x.x\mathbf{c}_1\Omega\mathbf{c}_1,\\ \lambda x.x\Omega\mathbf{c}_2\mathbf{c}_2\end{array}\right\}$ is not separable, although each proper subset is.

9. Definition. A set $\mathcal{X} \subseteq \Lambda^{\emptyset}$ is called an *adequate numeral system* iff for some combinators $O, S, Z_?, P$ one has

(i) $\mathcal{X} = \{S^n O \mid n \in \mathbb{N}\}$.

(ii) $P(S^{n+1}O) = S^n O$.

(iii) $Z_? O = \mathsf{U}^2_1$ & $Z_?(S^{n+1}O) = \mathsf{U}^2_2$.

Then all partial computable functions can be represented on $\mathcal{X}$, see Barendregt [1984] Proposition 6.4.3 and the remark following.

10. Definition. Let $\mathcal{V}$ be a set of combinators.

(i) $\mathcal{V}$ is called *RE (recursively enumerable)* if the set $\#\mathcal{V} = \{\#M \mid M \in \mathcal{V}\}$ is an RE set of natural numbers.

(ii) $\mathcal{V}$ is *closed under β-conversion* iff $\forall M, N.(M \in \mathcal{V}$ & $M =_\beta N) \Rightarrow N \in \mathcal{V}$.

(iii) $\mathcal{V}^\beta = \{N \mid \exists M \in \mathcal{V}.M =_\beta N\}$, the *$\beta$-closure of $\mathcal{V}$*.

(iv) $\mathcal{V}$ is a *V-set* iff $\mathcal{V}$ is RE and closed under β-conversion. These sets are the closed sets in the Visser topology, see Visser [1980] or Barendregt [1984], Definition 17.1.12.

11. Lemma. $\mathcal{V} \subseteq \Lambda^{\emptyset}$ be non-empty . Then $\mathcal{V}$ is a V-set iff for some $F \in \Lambda^{\emptyset}$ one has $\mathcal{V} = \{F\mathbf{c}_n \mid n \in \mathbb{N}\}^\beta$.

Proof. ($\Rightarrow$) By assumption $\#\mathcal{V}$ is non-empty and RE. Then $\#\mathcal{V} =$ $\{g(n) \mid n \in \mathbb{N}\}$ for some total computable function g. Then

$$V = \{\mathsf{E}\mathbf{c}_n \mid n \in \#\mathcal{V}\}^\beta.$$

Let $G \in \Lambda^{\emptyset}$ lambda define g. Then

$$\begin{aligned}\mathcal{V} &= \{\mathsf{E}\mathbf{c}_{g(n)} \mid n \in \mathbb{N}\}^\beta,\\ &= \{\mathsf{E}(G\mathbf{c}_n) \mid n \in \mathbb{N}\}^\beta,\\ &= \{F\mathbf{c}_n \mid n \in \mathbb{N}\}^\beta, \qquad \text{with } F = \mathsf{E} \circ G.\end{aligned}$$

($\Leftarrow$) $M \in \{F\mathbf{c}_n \mid n \in \mathbb{N}\}^\beta$ iff $\exists n.N =_\beta F\mathbf{c}_n$, which is RE. Clearly this set is closed under β-conversion.

In the present paper the following will be proved. See 13(iv) for the definition of morphisms. Some of these results have been proved in Ronchi della Rocca [1981] under stronger hypotheses: (i) $\Leftrightarrow$ (iii), (v) $\Rightarrow$ (i).

12. Theorem. Suppose that $\mathcal{V}$ is an infinite RE set (after coding) of combinators closed under β-conversion. Then the following are equivalent.

(i) $\mathcal{V}$ is an adequate numeral system.

(ii) For every morphism Φ with $\mathrm{dom}(\Phi) \subseteq \mathcal{V}$ there is an $F \in \Lambda^{\emptyset}$ such that

$$\forall M \in \mathcal{V}. \Phi(M) = FM.$$

(iii) There exists a combinator Δ such that for all $M, N \in \mathcal{V}$

$$\begin{aligned} \Delta MN &= \mathsf{U}^2_1, \text{ if } M = N; \\ &= \mathsf{U}^2_2, \text{ else.} \end{aligned}$$

(iv) There is a morphism Φ with $\mathrm{dom}(\Phi) \subseteq V$ such that for all $M, N \in \mathcal{V}$

$$\begin{aligned} \Phi(M)N &= \mathsf{U}^2_1, \text{ if } M = N, \\ &= \mathsf{U}^2_2, \text{ else.} \end{aligned}$$

(v) $\mathcal{V}$ is separable.

One way to think of Böhm's theorem is that it says that separating morphisms can be realized by terms.

2. Preliminaries

13. Definition. (i) Let $\# : \Lambda^{\emptyset} \to \mathbb{N}$ be an effective surjective Gödel numbering of combinators. We write $\ulcorner M \urcorner$ for $\mathbf{c}_{\#M}$.

(ii) There is an inverse E, called *Kleene's enumerator*, such that $\mathsf{E}\ulcorner M \urcorner = M$, for all combinators M, see Barendregt [1984] Theorem 8.1.6.

(iii) For $m, n \in \mathbb{N}$ we write $m \sim n \iff \mathsf{E}\mathbf{c}_m = \mathsf{E}\mathbf{c}_n$.

(iv) A *(partial Ershov) morphism* $\Phi : \Lambda^{\emptyset}/= \to \Lambda^{\emptyset}/=$ is a partial map such that for some partial computable function $\varphi : \mathbb{N} \to \mathbb{N}$ and all combinators M

$$\Phi(M) \cong \mathsf{E}(\mathbf{c}_{\varphi(\#M)}),$$

where $P \cong Q$ means that if one of the two expressions P, Q is defined, then so is the other and $P = Q$. This is implied by $\#\Phi(M) \simeq \varphi(\#M)$, with a similar meaning for $\simeq$: for expressions e_1, e_2 involving partial functions we define

$$e_1 \simeq e_2 \iff [e_1\downarrow \iff e_2\downarrow] \; \& \; [e_1\downarrow \Rightarrow e_1 \sim e_2].$$

See the last section for a discussion about the origin of Ershov morphisms.

(v) The notion of morphism generalizes naturally to binary maps. $\Phi : (\Lambda^{\emptyset})^2 \to \Lambda^{\emptyset}$ is a morphism if for some binary partial computable φ one has

$$\Phi(M, N) \cong E\mathbf{c}_{\varphi(\#M, \#N)}.$$

(vi) We write $\Phi(M)\downarrow, \varphi(m)\downarrow$ for convergence of the partial functions (being defined); similarly $\Phi(M)\uparrow, \varphi(m)\uparrow$ for divergence (being undefined).

14. Lemma. A partial morphism Φ is completely determined by a partial computable φ such that

$$\begin{aligned} \Phi(M) &\cong \mathsf{E}\mathbf{c}_{\varphi(\#M)}; \\ n \sim m &\Rightarrow \varphi(n) \simeq \varphi(m). \end{aligned}$$

In this case Φ is called the morphism corresponding to φ.

PROOF. This is the defining property for morphisms. We emphasize here that if $M =_\beta N$ and $\Phi(M)\downarrow$, then $\Phi(M) = \Phi(N)$, hence $\varphi(\#M) \sim \varphi(\#N)$. ■

Although there are partial recursive functions that cannot be made total, this is not the case for partial morphisms, as shown in Statman [1999].

15. THEOREM (Morphism extension). Suppose that Φ is a partial morphism. Then there exists a total morphism F extending Φ.

PROOF. Let Φ be correspond to the partial computable function φ. Construct a combinator P such that

$$Px\ulcorner M\urcorner =_\beta \begin{cases} \mathsf{E}x\ulcorner N\urcorner & \text{if } N =_\beta M \;\&\; \#N < \#M \;\&\\ & N \text{ is the first such found in some enumeration}\\ & \text{of the beta converts of } M;\\ \mathsf{Ec}_{\varphi(\#M)} & \text{if } \varphi(\#M) \text{ converges before such } N \text{ is found.}\end{cases}$$

[If $\varphi(\#M)\uparrow$ and $\neg\exists N =_\beta M.(\#N < \#M)$, then the search continues forever; in that case $Px\ulcorner M\urcorner$ will be unspecified and can be arranged to be unsolvable.] By the second fixed-point theorem, see Barendregt [1984], Theorem 6.5.9, there is a combinator Q such that $P\ulcorner Q\urcorner = Q$. Then

$$Q\ulcorner M\urcorner =_\beta \begin{cases} Q\ulcorner N\urcorner & \text{if } N =_\beta M \;\&\; \#N < \#M \;\&\\ & N \text{ is the first such found in some enumeration}\\ & \text{of the beta converts of } M;\\ \mathsf{Ec}_{\varphi(\#M)} & \text{if } \varphi(\#M) \text{ converges before such } N \text{ is found.}\end{cases}$$

If $\varphi(M)\downarrow$, then, using Lemma 14, it can be seen that a typical computation for $Q\ulcorner M\urcorner$ is the following

$$\begin{aligned} Q\ulcorner M\urcorner = Q\ulcorner M_1\urcorner = \ldots = Q\ulcorner M_k\urcorner = \mathsf{Ec}_{\#\varphi(M_k)} =\\ = \Phi(M_k) = \ldots = \Phi(M_1) = \Phi(M),\end{aligned}$$

with

$$\begin{array}{rrcr} \#M > & \#M_1 > & \ldots > & \#M_k,\\ M =_\beta & M_1 =_\beta & \ldots =_\beta & M_k,\\ \#M \sim & \#M_1 \sim & \ldots \sim & \#M_k,\\ \varphi(\#M) \sim & \varphi(\#M_1) \sim & \ldots \sim & \varphi(\#M_k),\\ \Phi(M) = & \Phi(M_1) = & \ldots = & \Phi(M_k).\end{array}$$

Therefore one has

$$Q\ulcorner M\urcorner =_\beta \begin{cases} \Phi(N) & \text{if } \Phi(M)\downarrow;\\ Q\ulcorner M_0\urcorner & \text{else,}\end{cases}$$

where $M_0 =_\beta M$ with the smallest Gödel number and $Q\ulcorner M_0\urcorner$ is unsolvable.

Finally let f be defined by $f(\#M) = \#(Q\ulcorner M\urcorner)$. Then it is easy to see that

1. f is a total computable function;
2. if $M =_\beta N$, then $Q\ulcorner M\urcorner =_\beta Q\ulcorner N\urcorner$, hence $f(\#M) \sim f(\#N)$;
3. if $\varphi(\#M)\downarrow$, then $\varphi(\#M) \sim f(\#M)$.

Thus there is a total morphism F determined by f. Moreover F extends Φ: $FM = Q\ulcorner M \urcorner = \Phi(M)$, if the latter is defined. ■

16. Definition. Let $\mathcal{V} \subseteq \Lambda^{\emptyset}$. Then $\mathcal{V}$ is a *V-set* if it is RE and closed under β-conversion.

17. Definition. Let $\mathcal{V}$ be a V-set.

(i) A $\mathcal{V}$-morphism is a partial Ershov morphism whose domain includes $\mathcal{V}$

(ii) A $\mathcal{V}$-morphism f is $\mathcal{V}$-representable if there exists an $F \in \Lambda^{\emptyset}$ such that

$$\forall M \in \mathcal{V}. FM = f(M).$$

A similar definition holds for binary morphisms.

(iii) Δ is a *Church discriminator (or Church delta)* for $\mathcal{V}$ if for all $M, N \in \mathcal{V}$ one has

$$\begin{aligned} \Delta MN &= \mathsf{U}^2_1, \text{ if } M = N; \\ \Delta MN &= \mathsf{U}^2_2, \text{ if } M \neq N. \end{aligned}$$

(iv) Let $M \in \mathcal{V}$ and $\Phi = \Phi_M$ be a $\mathcal{V}$-morphism. Then Φ is a $\mathcal{V}$-*equality test* for M if for all $N \in \mathcal{V}$

$$\begin{aligned} \Phi(N) &= \mathsf{U}^2_1, \text{ if } M = N, \\ \Phi(N) &= \mathsf{U}^2_2, \text{ if } M \neq N. \end{aligned}$$

18. Lemma. Let $\mathcal{V}$ be a V-set. If every unary morphism on $\mathcal{V}$ is $\mathcal{V}$-representable, then the same is true for binary morphisms.

Proof. Given a binary morphism Φ, define

$$\Psi(M) := \Phi(M\,\mathsf{U}^2_1, M\,\mathsf{U}^2_2).$$

Let Ψ be $\mathcal{V}$-representable by ψ. Construct a binary partial computable function φ such that $\varphi(\#M, \#N) = \psi(\#\langle M, N\rangle)$. Then Φ is represented by φ:

$$\begin{aligned} \Phi(M, N) &= \Psi(\langle M, N\rangle) \\ &= \mathsf{Ec}_{\psi(\#\langle M,N\rangle)} \\ &= \mathsf{Ec}_{\varphi(\#M,\#N)}. \ ■ \end{aligned}$$

19. Fact. The following statements are equivalent for a V-set $\mathcal{V}$.

(i) There is a $\mathcal{V}$-morphism Φ such that

$$\begin{aligned} &\forall M, N \in \mathcal{V}. [\Phi(M) = \Phi(N) \ \Rightarrow \ M = N] \ \& \\ &\forall M \in \mathcal{V} \exists n \in \mathbb{N}. \Phi(M) = \mathbf{c}_n. \end{aligned}$$

(ii) $\{\langle M, N\rangle \mid M, N \in \mathcal{V} \ \& \ M \neq N\}$ is RE.

Hence if $\mathcal{V}$ is a separable V-set, then $\{\langle M, N\rangle \mid M, N \in \mathcal{V} \ \& \ M \neq N\}$ is RE.

3. Böhm's theorem for V-sets

20. THEOREM. For $\mathcal{V}$ an infinite V-set the following are equivalent.

(i) $\mathcal{V}$ is an adequate numeral system.

(ii) Every $\mathcal{V}$-morphism is $\mathcal{V}$-representable.

(iii) There is a Church discriminator for $\mathcal{V}$.

(iv) There is a $\mathcal{V}$-morphism Φ such that

$$\forall M \in \mathcal{V}. \Phi(M) \text{ is a } \mathcal{V}\text{-equality test for } M.$$

(v) $\mathcal{V}$ is separable.

PROOF. We shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Write $\mathbf{v}_n := S^n O$. By Barendregt [1984] Lemma 6.4.5 there exists an H such that

$$H(\mathbf{v}_n) = \mathbf{c}_n. \tag{1}$$

The function $q(n) = \#\mathbf{v}_n$ (reminiscent of quoting) is total computable, so by Barendregt [1984] Theorem 6.4.3, it is lambda definable w.r.t. $(V, S, P, O, Z_?)$ by, say, Q, i.e.

$$Q\mathbf{v}_n = \mathbf{v}_{q(n)} = \mathbf{v}_{\#\mathbf{v}_n}. \tag{2}$$

Now suppose Φ is a partial morphism whose domain contains the set $\mathcal{V}$. By definition there is a partial computable function φ, such that

$$\Phi(M) = \mathsf{E}\mathbf{c}_{\varphi(\#M)}. \tag{3}$$

This f is lambda definable on $(V, S, P, O, Z_?)$ by, say, F. This means that

$$F\mathbf{v}_n = \mathbf{v}_{\varphi(n)}. \tag{4}$$

Let E be Kleene's enumerator and set $J := \lambda x.\mathsf{E}(H(F(Qx)))$. Then

$$\begin{aligned} J(\mathbf{v}_n) &= E(H(F(Q(\mathbf{v}_n)))) \\ &= E(H(F(\mathbf{v}_{\#\mathbf{v}_n}))), && \text{by (2)}, \\ &= E(H(\mathbf{v}_{\varphi(\#\mathbf{v}_n)})), && \text{by (4)}, \\ &= E\mathbf{c}_{\varphi(\#\mathbf{v}_n)}, && \text{by (1)}, \\ &= \Phi(\mathbf{v}_n), && \text{by (3)}. \end{aligned}$$

Thus Φ is $\mathcal{V}$-represented by J.

(ii) $\Rightarrow$ (iii). Let $M, N \in \mathcal{V}$ with $M \neq N$. Define a partial morphism Φ by

$$\begin{aligned} L = M &\Rightarrow \Phi(L) = \mathsf{U}^2_1 \\ L = N &\Rightarrow \Phi(L) = \mathsf{U}^2_2. \end{aligned}$$

Then this partial morphism extends to a total morphism by Theorem 15, which is *a fortiori* a $\mathcal{V}$-morphism. By hypothesis this morphism is $\mathcal{V}$-representable and thus for some F one has

$$FM = \mathsf{U}^2_1 \;\&\; FN = \mathsf{U}^2_2.$$

Hence, in particular, the set $\{(M,N)\in\mathcal{V}^2 \mid M\neq N\}$ is RE. Thus the partial function Φ on $\mathcal{V}^2$ such that

$$\begin{aligned}\Phi(M,N) &= \mathsf{U}^2_1, \text{ if } M=N,\\ \Phi(M,N) &= \mathsf{U}^2_2, \text{ if } M\neq N,\end{aligned}$$

is a $\mathcal{V}$-morphism, which by hypothesis and Lemma 18 is representable. In conclusion, there is a Church discriminator for $\mathcal{V}$.

(iii) $\Rightarrow$ (iv). Immediate, taking $\Phi(N) := \Delta N$.

(iv) $\Rightarrow$ (v). Let Φ be as in (iv) correspond to the partial computable φ which is λ-defined by F. Let G be an enumeration of $\mathcal{V}$, i.e. $\mathcal{V}=\{G\mathbf{c}_n \mid n\in\mathbb{N}\}$, possible by the definition of V-set. We want to define D such that

$$\begin{aligned}DM &= \mathbf{c}_{\mu y.[G\mathbf{c}_y=M]} \qquad (5)\\ &= \mathbf{c}_{\mu y.[\Phi((G\mathbf{c}_y))M=\mathsf{U}^2_1]}\\ &= \mathbf{c}_{\mu y.[\mathsf{E}(F(G\mathbf{c}_y))M=\mathsf{U}^2_2]}.\end{aligned}$$

Now the right-hand-side can be defined as $HM\mathbf{c}_0$ if

$$HM = \lambda y.\mathsf{E}(F(Gy))My(HM(S^+y)),$$

where $S^+$is the successor for the Church numerals. This is the case if we take

$$H := \mathsf{Y}(\lambda hmy.\mathsf{E}(F(Gy))my(hm(S^+y)),$$

where Y is the Fixed-point combinator. Then (v) via $D := \lambda m.Hm\mathbf{c}_0$, by (5).

(v) $\Rightarrow$ (i). Let $\mathcal{V}=\{F\mathbf{c}_e \mid e\in\mathbb{N}\}$ and let $d:\mathcal{V}\to\mathbb{N}$ be an injection definable by lambda term D . Then the set $\{(M,N)\in\mathcal{V}^2 \mid M\neq N\}$ is (after coding) RE. Define

$$\begin{aligned}O &:= F\mathbf{c}_0\\ S &:= \lambda x.F(\mu y.[\forall z \leq (\mu n.[F\mathbf{c}_n = x]).Fy \neq Fz])\\ P &:= \lambda x.F(\mu z.[S(Fz)=x])\\ Z_? &:= \text{Eq } (DO)(Dx),\end{aligned}$$

where Eq is the test for equality on the Church numerals. Then

$$\mathcal{V}=\{S^nO \mid n\in\mathbb{N}\},\ P(S^{n+1}O)=S^nO,\ Z_?O=\mathsf{U}^2_1 \text{ and } Z_?(S^{n+1}O)=\mathsf{U}^2_2. \blacksquare$$

21. Corollary. Not every total Ershov morphism on $\Lambda^{\emptyset}$ is representable.

Proof. Indeed, by Theorem 3.1 it would follow that there is a Church discriminator Δ for $\Lambda^{\emptyset}$, but then $\lambda x.\Delta x\mathsf{U}^2_2$ has no fixed-point, contradiction. $\blacksquare$

We end with some examples showing that there are various ways in which one can have equality tests.

22. Proposition. (i) Let $\mathcal{V}$ be a V-set with a $\mathcal{V}$-equality test for each member of $\mathcal{V}$ (but not uniformly so). This means that for all $M \in \mathcal{V}$ there exists a $\mathcal{V}$-morphism Φ_M such that for all $N \in \mathcal{V}$

$$\begin{aligned} \Phi_M(N) &= \mathsf{U}^2_1, \text{ if } M = N; \\ &= \mathsf{U}^2_2, \text{ else.} \end{aligned}$$

Then it does not follow that there is a Church's discriminator for $\mathcal{V}$.

(ii) Let $\mathcal{V}$ be a V-set with a $\mathcal{V}$-equality test for each member of $\mathcal{V}$ that is $\mathcal{V}$-representable. This means that for all $M \in \mathcal{V}$ there exists an $F_M \in \Lambda^{\emptyset}$ such that for all $N \in \mathcal{V}$

$$\begin{aligned} F_M N &= \mathsf{U}^2_1, \text{if } M = N; \\ &= \mathsf{U}^2_2, \text{else.} \end{aligned}$$

Suppose moreover that $\{\langle M, N\rangle \mid M, N \in \mathcal{V} \ \& \ M \neq N\}$ is RE. Even then $\mathcal{V}$ does not necessarily have a Church discriminator.

Proof. (i) We will construct $\mathcal{V}$ with $\{\langle M, N\rangle \mid M, N \in \mathcal{V} \ \& \ M \neq N\}$ is not RE. This suffices by Fact 19. Define the following partial computable function

$$\begin{aligned} \psi(e, x) &= x, && \text{if } \{e\}(0) \text{ converges in exactly } x \text{ steps,} \\ &= 1 + \psi(e, x+1), && \text{else.} \end{aligned}$$

Here $\{e\}(x)$ is the result of the partial computable function with code (program) e and input x. By Kleene's theorem ψ is represented by a lambda term G and set $F := \lambda n.Gn\mathbf{c}_0$. Thus $F\mathbf{c}_e$ has finite Böhm tree $\mathrm{BT}(\mathbf{c}_n)$ if $\{e\}(0)$ converges in n steps and it can be arranged that otherwise $F\mathbf{c}_e$ has the infinite Böhm tree

$$\infty := \quad \begin{array}{c} \lambda xy.x \\ | \\ x \\ | \\ x \\ | \\ \cdots \end{array}$$

For each e there are many e^* such that $\{e^*\} = \{e\}$. In case $e(0)\uparrow$ for such e, e^*, one has

$$e \neq e^* \ \Rightarrow \ F\mathbf{c}_e \neq F\mathbf{c}_{e^*}, \tag{1}$$

even if $\mathrm{BT}(F\mathbf{c}_e) = \mathrm{BT}(F\mathbf{c}_{e^*})$ (the difference is 'pushed to infinity', hence the trees are equal). Take $\mathcal{V} = \{F\mathbf{c}_e \mid e \in \mathbb{N}\}/=_\beta$. We show that this V-set works. For each fixed combinator $M \in \mathcal{V}$, say $M = F\mathbf{c}_e$, we have to decide whether for a given combinator N one has $N = M$.

Case 1. $\{e\}(0)\downarrow$ in n steps. Then $M = \mathbf{c}_n$. Given $N \in \mathcal{V}$ we develop its Böhm tree, which will be one of $\{\mathrm{BT}(\mathbf{c}_0), \mathrm{BT}(\mathbf{c}_1), \ldots; \infty\}$. If $\mathrm{BT}(N) = \mathbf{c}_k$ with $k < n$, then $N \neq M$ and the output should be (the Gödel number of) U^2_2. If

$\mathrm{BT}(N) = \mathbf{c}_n$, then the output should be U^2_1. Finally, if $\mathrm{BT}(N)$ keeps growing beyond $\mathrm{BT}(\mathbf{c}_n)$, then the output is again U^2_2.

Case 2. $\{e\}(0)\uparrow$. Then for $N \in \mathcal{V}$ one can check $N = M$ as follows. Find an e' such that $N = F\mathbf{c}_{e'}$. Then $N = M \;\Leftrightarrow\; F\mathbf{c}_e \neq F\mathbf{c}_{e^*} \;\Leftrightarrow\; e' = e$, by (1).

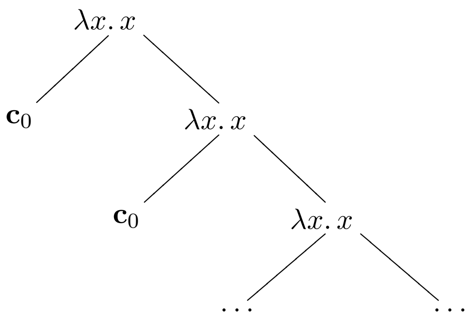
(ii) We will construct such a $\mathcal{V}$. First let T be Kleene's T predicate, i.e. $T(e,x,y)$ iff y is the code of a terminating computation for $\{e\}(x)$, the result of the partial computable function with program e on input x. Define the following total computable function.

$$\begin{aligned} f(e,n) &= 0, && \text{if } n \leq e \;\&\; \neg\exists k \leq n.T(e,0,k);\\ &= 1, && \text{if } n \leq e \;\&\; \exists k \leq n.T(e,0,k);\\ &= f(e,e), && \text{else.} \end{aligned}$$

Write $\mathbf{e}_n := \mathbf{c}_{f(e,n)}$. Define the following lambda terms for $n \in \mathbb{N}$:

$$n_\infty = \mathsf{Y}(\lambda p.\langle \mathbf{c}_n, p\rangle).$$

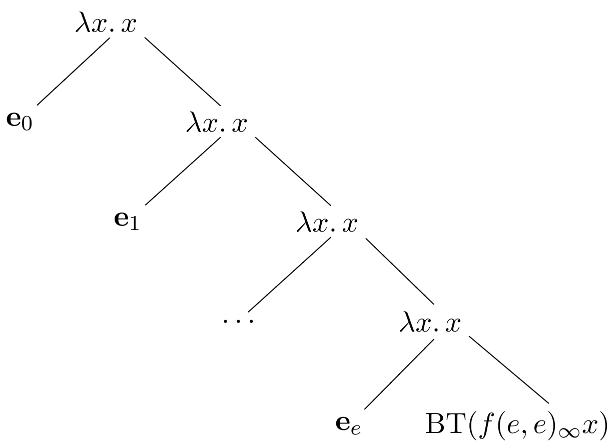
Then $n_\infty = \langle \mathbf{c}_n, n_\infty\rangle$ and e.g. 0_∞ has as Böhm-tree



It is not hard to construct terms M_e such that

$$\begin{aligned} M_e &:= H\mathbf{c}_e\mathbf{c}_0;\\ H\mathbf{c}_e\mathbf{c}_n &:= \text{If } (n \leq e \;\&\; \neg T(e,0,n)) \text{ then } \langle \mathbf{c}_0, H\mathbf{c}_e\mathbf{c}_{n+1}\rangle\\ &\quad\;\; \text{else [If } (n \leq e \;\&\; T(e,0,n)) \text{ then } 1_\infty \text{ else } 0_\infty]. \end{aligned}$$

having as Böhm-Trees



This tree has as its short left branches the trees of $\mathbf{c}_0, \mathbf{c}_0, \dots$ ad infinitum, unless a computation of $\{e\}(0)$ converges and this happens in $k \leq e$ steps, in which case the $k+p$-th branches become $\mathbf{c}_1$ for all p. Let

$$\mathcal{V} = \{M_e \mid e \in \mathbb{N}\}.$$

One has the following.

(1) $\forall k. f(e,k) = f(e',k) \Leftrightarrow \forall k \leq \max\{e,e'\}. f(e,k) = f(e',k)$
$\Leftrightarrow M_e =_\beta M_{e'}$.

(2) $\{(e,e') \mid M_e \neq_\beta M_{e'}\}$ is RE.

(3) $\forall M \in \mathcal{V} \exists D_M \in \Lambda^{\emptyset} \forall N \in \mathcal{V}.\ D_M N = \mathsf{U}^2_1$, if $M = N$,

$$D_M N = \mathsf{U}^2_2, \text{ else.}$$

Given M one can determine the e such that $M := M_e$. Then $M = N$, for $N \in \mathcal{V}$, iff up to level e one has $\mathrm{BT}(M) = \mathrm{BT}(N)$.

(4) $\neg \exists \Delta \in \Lambda^{\emptyset} \forall M, N \in \mathcal{V}. \Delta M N = \mathsf{U}^2_1$, if $M = N$,

$$\Delta M N = \mathsf{U}^2_2, \text{ else.}$$

If such a Δ would exist, then by the continuity of application with respect to the tree topology, see Barendregt [1984], Theorem 14.3.22, the value of $\Delta M N$ is determined by a fixed finite approximation of the Böhm-trees of $M, N \in \mathcal{V}$. But there are always terms in $\mathcal{V}$ that start to be different at deeper levels. ∎

4. Discussion

1. In Ershov [1973,1975,1977], see also Visser [1980], the notion of *numbered set* is intyroduced. This is a pair (S, γ) with $\gamma : \mathbb{N} \to S$ a surjection. An n such that $\gamma(n) = s$ is called a *code* of s. A (Ershove) morphism between numbered sets $(S_1, \gamma_1), (S_2, \gamma_2)$ is a map $\mu : S_1 \to S_2$ such that for some total computable function $f : \mathbb{N} \to \mathbb{N}$ one has

   $$\forall n \in \mathbb{N}. \mu(\gamma_1(n)) = \gamma_2(f(n)).$$

   That is, a morphism is determined by a computable map on the codes. Given a numbered set (S, γ) one defines an equivalence relation on $\mathbb{N}$ by $n \sim m \Leftrightarrow \gamma(n) = \gamma(m)$. This numbered set is called *pre-complete* iff every partial computable function on $\mathbb{N}$ can be made total modulo $\sim$:

   $$\forall \psi \text{ partial computable} \, \exists f \text{ total computable} \, \forall n \in \mathbb{N}. \psi(n)\downarrow \ \Rightarrow\ \psi(n) \sim f(n).$$

   One of the first results in the the theory of numbered sets is that each morphism from a pre-complete numbered set to itself has a fixed point. An example of a pre-complete numbered set is $(\Lambda^{\emptyset}, \gamma_{\mathsf{E}})$, with $\gamma_{\mathsf{E}}(n) = \mathsf{E}\mathbf{c}_n$.
2. There are numeral systems on which all partial computable functions can be represented, without there being a test for zero $Z_?$, see Intrigila [1994] or Barendsen [1991]. Such numeral systems are not separable.
3. In Barendregt [1984] one uses the notation $\ulcorner M \urcorner = \ulcorner \# M \urcorner$, for M a lambda term. Here one uses a different system of numerals, denoted by $\ulcorner n \urcorner$, for $n \in \mathbb{N}$. This does not matter, as the $\mathbf{c}_n$ and the $\ulcorner n \urcorner$ are equivalent in the sense that for some combinators G, H one has $G\mathbf{c}_n = \ulcorner n \urcorner$ & $H \ulcorner n \urcorner = \mathbf{c}_n$.

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