Self-interpretation in lambda calculus

More on data types

A data type D is a set with some operations (functions) on it.

An k-ary operation is a function $f : D^k \longrightarrow D$.

Thereby a 0-ary operation $c : D^0 \longrightarrow D$ is identified with a $c{\in}D$.

A datatype is determined by its operations on D:

$$\begin{array}{rcl} c_1, \ldots, c_{k_0} & : & D^0 \longrightarrow D \; = \; D \\ f^1_1, \ldots, f^1_{k_1} & : & D^1 \longrightarrow D \\ f^2_1, \ldots, f^2_{k_2} & : & D^2 \longrightarrow D \\ & \ldots & \end{array}$$

$\texttt{Nat}$ has $\texttt{z} : \texttt{Nat},\ \texttt{s} : \texttt{Nat} \longrightarrow \texttt{Nat}$.

$\texttt{Tree}$ has $\texttt{l} : \texttt{Tree}, \texttt{p} : \texttt{Tree}^2 \longrightarrow \texttt{Tree}$

Packing and unpacking λ-terms

Given $M_1, \ldots, M_k$, define

$$\langle M_1, \ldots, M_k \rangle \;:=\; \lambda z.zM_1 \ldots M_k$$

Define U_i^k, with $1 \le i \le k$ by

$$U_i^k \;:=\; \lambda x_1 \ldots x_k.x_i$$

Then

$$U_i^k M_1 \ldots M_k \;=\; M_i$$
$$\langle M_1, \ldots, M_k \rangle U_i^k \;=\; U_i^k M_1 \ldots M_k \;=\; M_i$$

Note that $\mathsf{K} = \mathsf{U}_1^2$

Second translation (Böhm-Piperno-Guerini)

Consider the data type D with

$$c : D,\ f : D \to D,\ g : D^2 \to D$$

The second coding (also denoted by $\ulcorner t \urcorner$) is

$$\begin{array}{rcl} \ulcorner c \urcorner & = & \lambda e.eU_1^3 e \\ \ulcorner f(t) \urcorner & = & \lambda e.eU_2^3 \ulcorner t \urcorner e \\ \ulcorner g(t_1, t_2) \urcorner & = & \lambda e.eU_3^3 \ulcorner t_1 \urcorner \ulcorner t_2 \urcorner e \end{array}$$

PROPOSITION. There are lambda terms F, G such that

$$\begin{array}{rcl} F \ulcorner t \urcorner & = & \ulcorner f(t) \urcorner \\ G \ulcorner t_1 \urcorner \ulcorner t_2 \urcorner & = & \ulcorner g(t_1, t_2) \urcorner \end{array}$$

PROOF. Take

$$\begin{array}{rcl} F & := & \lambda te.eU_2^3 te \\ G & := & \lambda t_1 t_2 e.eU_3^3 t_1 t_2 e. \blacksquare \end{array}$$

Recursion

THEOREM. Given $A_1, A_2, A_3 \in \Lambda$ there is an $H \in \Lambda$ such that

$$\begin{array}{rcccl} H\ulcorner c\urcorner & = & & = & A_1 H \\ H(F\ulcorner t\urcorner) & = & H(\ulcorner f(t)\urcorner) & = & A_2\ulcorner t\urcorner H \\ H(G\ulcorner t_1\urcorner\ulcorner t2\urcorner) & = & H(\ulcorner g(t_1,t_2)\urcorner) & = & A_3\ulcorner t_1\urcorner\ulcorner t_2\urcorner H \end{array}$$

PROOF. Try $H = \langle\langle B_1, B_2, B_3\rangle\rangle$.

$$\begin{array}{rcll} H\ulcorner c\urcorner & = & \langle\langle B_1, B_2, B_3\rangle\rangle\ulcorner c\urcorner & \\ & = & \ulcorner c\urcorner\langle B_1, B_2, B_3\rangle & \\ & = & \langle B_1, B_2, B_3\rangle U_1^3\langle B_1, B_2, B_3\rangle & \\ & = & B_1\langle B_1, B_2, B_3\rangle & \\ & = & A_1\langle\langle B_1, B_2, B_3\rangle\rangle, & \text{if } B_1 := \lambda z.A_1\langle z\rangle. \\ & = & A_1 H & \\ H\ulcorner f(t)\urcorner & = & \langle B_1, B_2, B_3\rangle U_2^3\ulcorner t\urcorner\langle B_1, B_2, B_3\rangle & \\ & = & B_2\ulcorner t\urcorner\langle B_1, B_2, B_3\rangle & \\ & = & A_2\ulcorner t\urcorner H, & \text{if } B_2 := \lambda tz.A_2 t\langle z\rangle. \\ H\ulcorner g(t_1,t_2)\urcorner & = & A_3\ulcorner t_1\urcorner\ulcorner t_2\urcorner H, & \text{if } B_3 := \lambda t_1 t_2 z.A_3 t_1 t_2\langle z\rangle.\ \blacksquare \end{array}$$

Data type for coding lambda terms

Consider the data type

$$\begin{aligned} \texttt{var}: &\ D \to D \\ \texttt{app}: &\ D \to D \to D \\ \texttt{abs}: &\ D \to D \end{aligned}$$

Define `Var`, `App`, `Abs` as follows

$$\begin{aligned} \texttt{Var} &:= \lambda xe.eU_1^3xe \\ \texttt{App} &:= \lambda xye.eU_2^3xye \\ \texttt{Abs} &:= \lambda xe.eU_3^3xe \end{aligned}$$

Coding lambda terms $M \leadsto \ulcorner M \urcorner$ (Mogensen)

$$\begin{aligned} \ulcorner x \urcorner &:= \texttt{Var}\ x \\ \ulcorner MN \urcorner &:= \texttt{App} \ulcorner M \urcorner \ulcorner N \urcorner \\ \ulcorner \lambda x.M \urcorner &:= \texttt{Abs}\ (\lambda x.\ulcorner M \urcorner) \end{aligned}$$

Self-interpretation

THEOREM. There exists a λ-term E such that for all $M \in \Lambda$

$$\mathsf{E}\ulcorner M \urcorner = M$$

PROOF. By recursion we can find an E such that

$$\begin{array}{rcl} \mathsf{E}(\mathtt{Var}\ x) & = & x \\ \mathsf{E}(\mathtt{App}\ m\ n) & = & \mathsf{E}m(\mathsf{E}n) \\ \mathsf{E}(\mathtt{Abs}\ m) & = & \lambda x.\mathsf{E}(mx) \end{array}$$

Then

$$\begin{array}{rcllcl} \mathsf{E}(\ulcorner x \urcorner) & = & \mathsf{E}(\mathtt{Var}\ x) & = & x & \\ \mathsf{E}(\ulcorner MN \urcorner) & = & \mathsf{E}(\mathtt{App}\ulcorner M \urcorner \ulcorner N \urcorner) & = & \mathsf{E}\ulcorner M \urcorner(\mathsf{E}\ulcorner N \urcorner) & = & MN \\ \mathsf{E}(\ulcorner \lambda x.M \urcorner) & = & \mathsf{E}(\mathtt{Abs}(\lambda x.\ulcorner M \urcorner)) & = & \lambda x.\mathsf{E}\ulcorner M \urcorner & = & \lambda x.M. \end{array}$$

Filling in the details of E one has (writing $\mathsf{C} := \lambda xyz.xzy$)

$$\mathsf{E} = \langle\langle \mathsf{K}, \mathsf{S}, \mathsf{C} \rangle\rangle. \qquad \blacksquare$$

Application 1

If you see someone coming out of ‘arrivals’ in an airport,
you cannot determine where he or she comes from.

Similarly, there is no F such that for all $X, Y \in \Lambda$

$$F(XY) = X$$

PROPOSITION. There exists an $F_i \in \Lambda$, $i \in \{1, 2\}$ such that

$$F_i \ulcorner X_1 X_2 \urcorner = \ulcorner X_i \urcorner.$$

PROOF. We do this for $i = 1$. By recursion there exists F_1 s.t.

$$F_1(\texttt{App}\ x_1\ x_2) = A_2 x_1 x_2 F_1 = x_1, \text{ taking } A_2 = \mathsf{U}_2^3.$$

This suffices. ■

Second fixed point theorem*

LEMMA. There exists a term $\mathtt{Num}\in\Lambda$ such that for all $M\in\Lambda$

$$\mathtt{Num}\ulcorner M\urcorner =_\beta \ulcorner\ulcorner M\urcorner\urcorner$$

PROOF. Use recursion for the lambda calculus data type with

$$\begin{aligned} A_1 x N &= \mathtt{App}\ulcorner\mathtt{Var}\urcorner(\mathtt{Var}\ x) \\ A_2 m n N &= \mathtt{App}(\mathtt{App}\ulcorner\mathtt{App}\urcorner(Nm))(Nn) \\ A_3 m N &= \mathtt{App}\ulcorner\mathtt{Abs}\urcorner(\mathtt{Abs}(\lambda x.N(mx))) \end{aligned}$$

SECOND FIXED POINT THEOREM. For all F there is an X with

$$F\ulcorner X\urcorner =_\beta X$$

PROOF. Let $W := \lambda z.F(\mathtt{App}\ z(\mathtt{Num}\ z))$ and $X := W\ulcorner W\urcorner$. Then

$$\begin{aligned} X &= W\ulcorner W\urcorner \\ &= F(\mathtt{App}\ulcorner W\urcorner(\mathtt{Num}\ulcorner W\urcorner) = F(\mathtt{App}\ulcorner W\urcorner\ulcorner\ulcorner W\urcorner\urcorner) \\ &= F\ulcorner W\ulcorner W\urcorner\urcorner = F\ulcorner X\urcorner. \blacksquare \end{aligned}$$

Application 2*

For a given T there exists a program P such that

$$\begin{aligned} P\mathbf{c}_k &= \mathbf{c}_{k+1}, && \text{if } k \text{ is even,} \\ &= T\ulcorner P \urcorner \mathbf{c}_k, && \text{otherwise.} \end{aligned}$$