Extra exercises Lecture 6

May 17, 2021

Exercise 1. For $G = (V, E)$, to decide if G has a *Hamilton circuit*, $\mathsf{Ham}(G)$, is known to be NP-complete. We haven't proved this in detail in the course, but you may assume it. (NB. A Hamilton circuit is a cyclic path in the graph that visits all vertices exactly once.)

A *nearly-Hamilton circuit* is a circuit in the graph G that has exactly one vertex occurring twice, and all others occurring exactly once. The problem $\mathsf{nearHam}(G)$ is to decide whether G has a nearly-Hamilton circuit.

(a) Give a graph $G = (V, E)$ that has a nearly-Hamilton circuit, but not a Hamilton circuit.

(b) Prove that $\mathsf{nearHam}$ is NP-complete.

Exercise 2. We define *modified* $\mathsf{3CNF}$ *formulas*, $m\mathsf{3CNF}$, as follows: $\varphi \in m\mathsf{3CNF}$ if $\varphi \in \mathsf{3CNF}$ and **every atom in φ occurs in at most 3 clauses**. The problem *modified* $\mathsf{3CNF\text{-}SAT}$, $m\mathsf{3CNF\text{-}SAT}(\varphi)$, is the problem of deciding for a formula $\varphi \in m\mathsf{3CNF}$ whether it is satisfiable or not.

(a) State precisely, in a formula, what $m\mathsf{3CNF\text{-}SAT}$ means and state precisely what needs to be proven for $m\mathsf{3CNF\text{-}SAT}$ to be NP-complete.

(b) Prove the following equivalence: $(\ell \vee A_i) \wedge (\ell \vee A_j)$ is satisfiable iff $(a \vee A_i) \wedge (a \vee A_j) \wedge (\ell \vee \neg a)$ is satisfiable, for a a fresh atom (where ℓ is a literal and A_i, A_j are arbitray formulas).

(c) Give a proof of the NP-completeness of $m\mathsf{3CNF\text{-}SAT}$.
Hint Use NP-completeness of $\mathsf{3CNF\text{-}SAT}$ and the equivalence in (b)

Exercise 3. For $C \subseteq \mathbb{Z}$, we consider $\mathsf{ILP}(C)$ which is a variant of the *integer linear programming problem*, ILP, that we have seen in the course. Given a finite set of inequalities of the form

$$a_1x_1 + \ldots + a_nx_n \leq c$$

where $c \in C$ and $a_1, \ldots, a_n \in \mathbb{Z}$, we ask if there are values $x_1, \ldots, x_n \in \mathbb{Z}$ such that all inequalities hold. In the course, we have seen that $\mathsf{ILP}(\mathbb{Z})$ is NP-complete.

(a) Give a set $C \subseteq \mathbb{Z}$, with C containing at least two elements, for which $\mathsf{ILP}(C)$ is not NP-complete.

(b) Show that for $C = \{-1, 1\}$, $\mathsf{ILP}(C)$ is NP-complete.
Hint Add a fresh variable x to each equation to transform an inequality $a_1x_1 + \ldots + a_nx_n \leq b$ to $a_1x_1 + \ldots + a_nx_n + ax \leq 1$; add additional equalities for x.