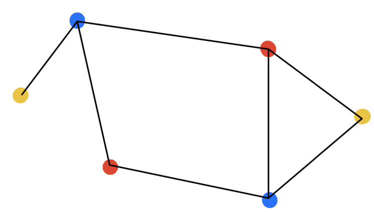
FINDING A 3-COLORING IS NP-COMPLETE

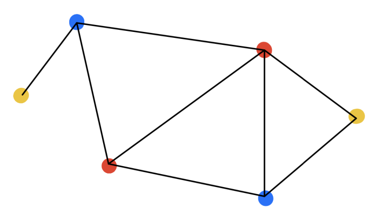
In this note, we show that 3Color is **NP**-complete. Let us start by recalling colorings of graphs

Definition 1. Let $G = (V, E)$ be a graph. A **3-coloring** of G is a function $c : V \to \{\mathbf{r}, \mathbf{b}, \mathbf{y}\}$ such that for every edge $e = \{v, w\} \in E$ we have $c(v) \neq c(w)$.

A coloring assigns to every vertex a color in such a way, that adjacent vertices have different colors. For example, the following is a 3-coloring



However, the following is not a 3-coloring, because two adjacent vertices have the same color.



We define 3Color to be the following decision problem:

Given a graph G, does G have a 3-coloring?

The goal of this note is to prove that 3Color is **NP**-complete. To do so, we must show the following two things:

- 3Color $\in$ **NP**.
- 3Color is **NP**-hard.

Let us start by proving 3Color $\in$ **NP**. So, we prove that we can check in polynomial time whether a function $c : V \to \{\mathbf{r}, \mathbf{b}, \mathbf{y}\}$ is a 3-coloring.

Proposition 2. 3Color $\in$ **NP**.

Proof. A certificate for this problem is a map $c : V \to \{\mathbf{r}, \mathbf{b}, \mathbf{y}\}$, and to verify this is a 3-coloring, one must check for every edge $e = \{v, w\}$ if G whether v and w are assigned a different color. Note that this can be done in linear time, and hence, 3Color $\in$ **NP**. □

It remains to show that 3Color is **NP**-hard. To prove this is the case, we construct the following reduction: 3CNF $\leq_P$ 3Color. Since we already know that 3CNF is **NP**-hard, this implies that 3Color is **NP**-hard.

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Suppose that we have a formula φ which is in 3-conjunctive normal form. Concretely, this means that φ has the following shape

$$\varphi = \bigwedge_{i=1}^{n} \varphi_i$$

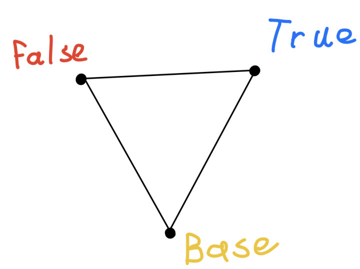
where each φ_i can be written as $\psi_{i,1} \vee \psi_{i,2} \vee \psi_{i,3}$ where each $\psi_{i,j}$ is a literal (*i.e.*, either an atom a_i or a negation $\neg a_i$ of some atom a_i). In addition, suppose that the atoms occurring in this formula are $a_1, \ldots, a_k$.

Our goal is to construct a graph G_φ such that G_φ has a 3-coloring if and only if φ is satisfiable. We also make sure that G_φ can be computed in polynomial time from φ. This graph has the following vertices:

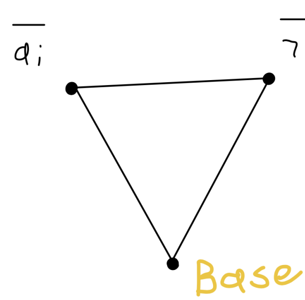
- We have nodes **True**, **False**, and **Base**.
- For each atom a_i we have nodes $\overline{a_i}$ and $\overline{\neg a_i}$.
- For each conjunct φ_i we have nodes $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, x_{i,5}$.

Note that since every $\psi_{i,j}$ is a literal, we can pick a node $\overline{a_k}$ or $\overline{\neg a_k}$ corresponding to that literal, and we denote this node by $\overline{\psi_{i,j}}$. There are the following edges:

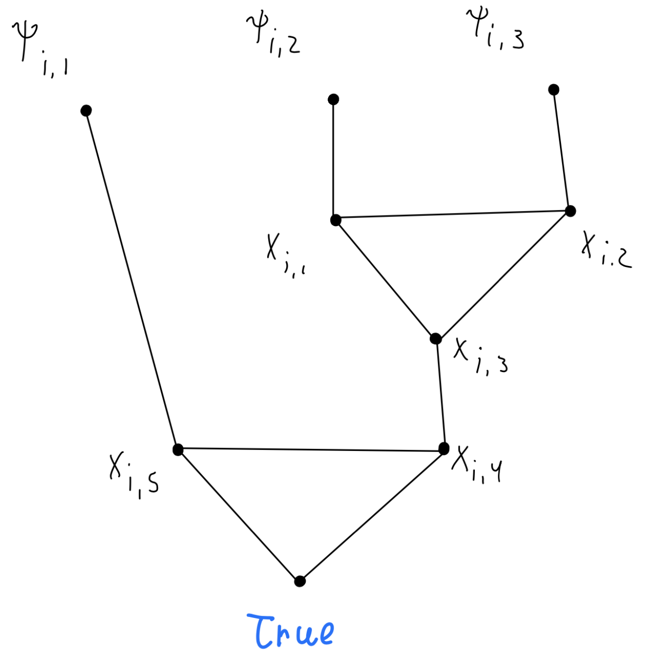
- There is a triangle of edges between **True**, **False**, and **Base** as follows



- For each atom a_i, we have a triangle as follows



- For every conjunct $\varphi_i = \psi_{i,1} \vee \psi_{i,2} \vee \psi_{i,3}$, we have edges as follows

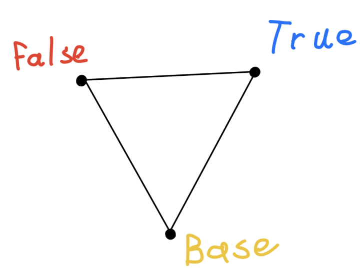


Before we prove that the 3-colorability of this graph coincides with the satisfiability of φ, let us think about what we can say about 3-colorings of this graph. Suppose that c is a coloring of G_φ. First, we observe that the nodes **True**, **False**, and **Base** get different colors.

Observation 1. *If we have a triangle of vertices, then each of those vertices is assigned a different color.*

Proof. This holds because adjacent vertices ae assigned different colors and because all of the vertices in the triangle are connected. □

In particular, every coloring c maps the nodes **True**, **False**, and **Base** to different values, since we have the following triangle of nodes.



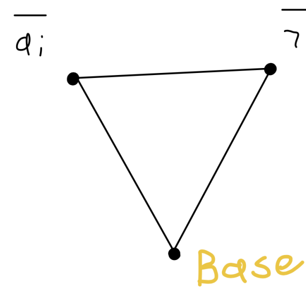
Because of Observation 1, we can assume without loss of generality that

$$c(\textbf{True}) = \textbf{b}, \quad c(\textbf{False}) = \textbf{r}, \quad c(\textbf{Base}) = \textbf{y}$$

Lemma 3. *We have the following inequalities:*

- $c(\overline{a_i}) \neq \mathbf{y}$ *and* $c(\overline{\neg a_i}) \neq \mathbf{y}$.
- $c(\overline{a_i}) \neq c(\overline{\neg a_i})$.

Proof. Again we use that adjacent vertices get different colors. The statement of both items follow from the following triangle

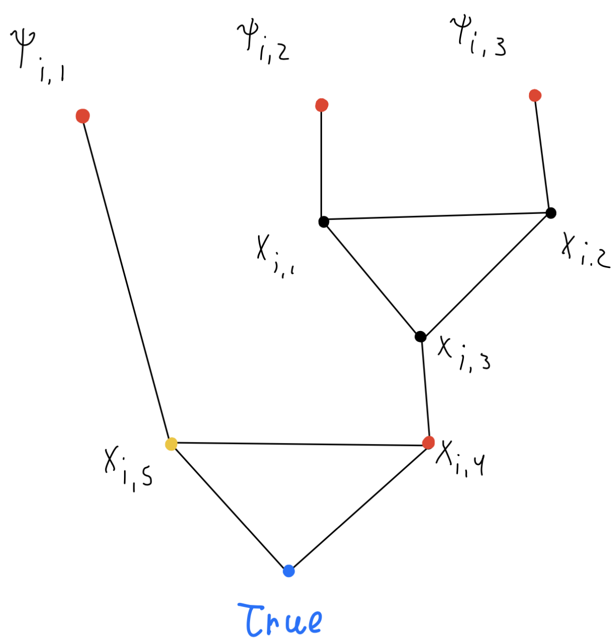


□

Now we can better understand how colorings relate to satisfiability. A coloring assigns to every literal either the color **r** or **b**, where the former represents that this literal is false while the latter represents this literal is true. In addition, since $c(\overline{a_i}) \neq c(\overline{\neg a_i})$, only one of these two gets mapped to **r** and the other to **b**.

Lemma 4. *Let φ_i be any conjunct of φ. Then it is not the case that $c(\psi_{i,1}) = c(\psi_{i,2}) = c(\psi_{i,3}) = \mathbf{r}$.*

Proof. Suppose, that we actually have that $c(\psi_{i,1}) = c(\psi_{i,2}) = c(\psi_{i,3}) = \mathbf{r}$. To conclude the lemma, we need to prove a contradiction. Using the fact that adjacent vertices have different colors, the coloring must look as follows.



Since $x_{i,5}$ is adjacent to both $\psi_{i,1}$ and **True**, it gets assigned the color **y**, and because $x_{i,4}$ is adjacent to both **True** and $x_{i,5}$, it is colored **r**. Note that $x_{i,1}$, $x_{i,2}$, and $x_{i,3}$ are all adjacent to each other. In addition, each of these three vertices is adjacent to a red one. Since there are only three colors, it is impossible to assign colors to $x_{i,1}$, $x_{i,2}$, and $x_{i,3}$ in such a way that we actually get a coloring. As such, we have reached the desired contradiction. □

Now we prove that 3Color is **NP**-hard.

Lemma 5. *If G_φ has a 3-coloring, then φ is satisfiable.*

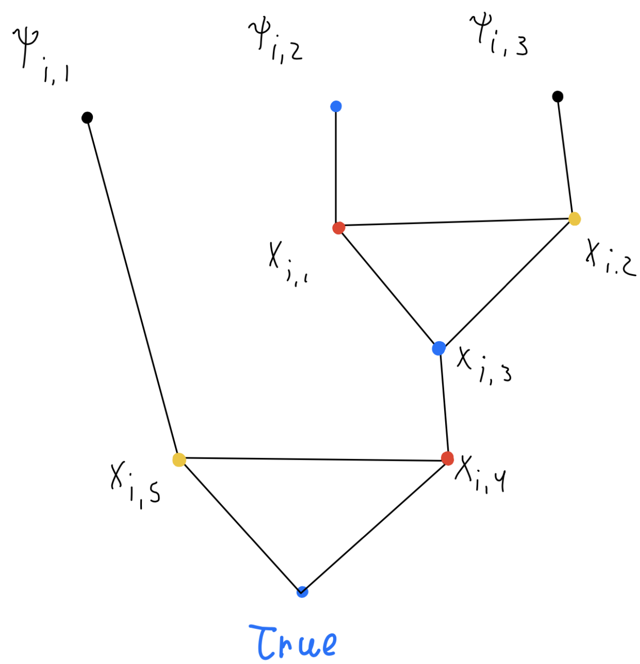
Proof. Note that from Lemma 3, we can map every atom to a truth value, and as such, we have a model m. Now we need to show that φ holds in m. Concretely, we need to show that for every φ_i, there is at least one literal that gets mapped to 1 by m. We showed in Lemma 4 that it is impossible that all the $\psi_{i,j}$ get mapped to false. Hence, φ holds in m □

To prove the converse, we use the same ideas but "in the opposite direction".

Lemma 6. *If φ is satisfiable, then G_φ has a 3-coloring.*

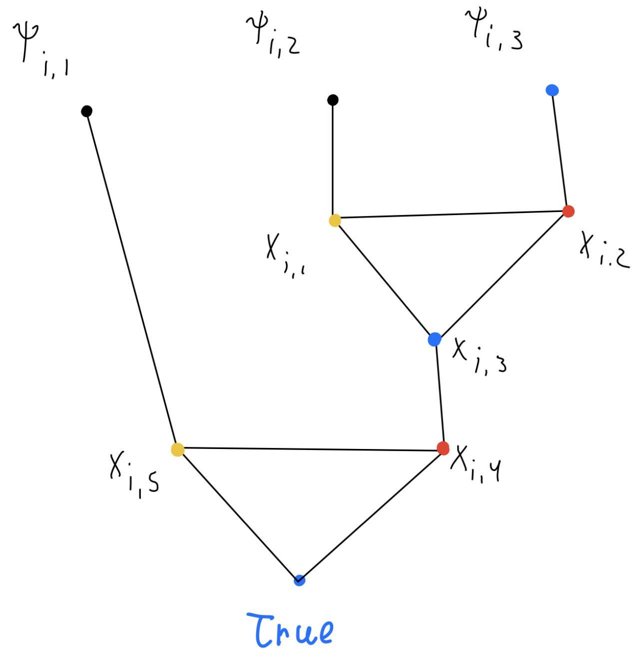
Proof. Suppose that φ is satisfiable, and let m be a model in which φ holds. Define the following 3-coloring on G_φ:

- We set
$$c(\textbf{True}) = \textbf{b}, \quad c(\textbf{False}) = \textbf{r}, \quad c(\textbf{Base}) = \textbf{y}$$
- If $m(a_i) = 1$, then we set $c(\overline{a_i}) = \textbf{b}$ and $c(\overline{\neg a_i}) = \textbf{r}$, while if $m(a_i) = 0$, we set $c(\overline{a_i}) = \textbf{r}$ and $c(\overline{\neg a_i}) = \textbf{b}$.
- Let $\varphi_i = \psi_{i,1} \vee \psi_{i,2} \vee \psi_{i,3}$ be a conjunct of φ. Note that $m(\varphi_i) = 1$, because φ holds under m and because φ is in conjunctive normal form. We consider three cases. If $m(\psi_{i,2}) = 1$, then we color the $x_{i,j}$ as follows



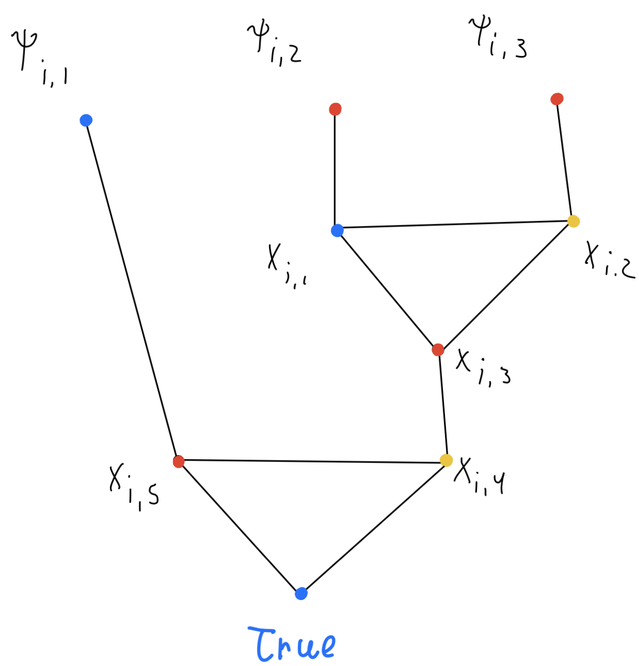
This coloring allows for $\psi_{i,1}$ and $\psi_{i,3}$ to be colored either red or blue.

If $m(\psi_{i,3}) = 1$, then we color the $x_{i,j}$ as follows



Again this coloring allows for $\psi_{i,1}$ and $\psi_{i,3}$ to be colored either red or blue.

Otherwise, we have $m(\psi_{i,2}) = m(\psi_{i,3}) = 0$. Since φ is satisfiable, we see that $m(\psi_{i,1}) = 1$, and then we assign the following colors



□

Now we can conclude the following.

Theorem 7. 3Color *is* $\mathbf{NP}$*-complete.*

Proof. To prove that 3Color is $\mathbf{NP}$-complete, we need to prove two things. First of all, we need to show that 3Color $\in \mathbf{NP}$, which was done in Proposition 2. Second of all, we had to prove that 3Color is $\mathbf{NP}$-hard. We did that by constructing a reduction from 3CNF to 3Color. Since we know that 3CNF is $\mathbf{NP}$-hard, this allows us to conclude that 3Color is $\mathbf{NP}$-hard. The proof that we indeed have such a reduction, was given in lemmas Lemmas 5 and 6. Note that G_φ can be computed in linear time, because the amount of vertices and edges in G_φ depends linearly on the size of φ. □

Definition 1 can be adapted to n-colorings for arbitrary n, and we can define the decision problem nColor analogously. This gives a large number of $\mathbf{NP}$-complete problems.

Exercise. The decision problem 4Color is $\mathbf{NP}$-complete.
Hint: show that 3Color $\leq_P$ 4Color.

Exercise. For every $n \geq 3$, the decision problem nColor is $\mathbf{NP}$-complete.
Hint: use induction and show that nColor $\leq_P$ $(n+1)$Color.