Exercises Complexity Theory

Lecture 1

April 8, 2024

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **April 15, 2023**, in Brightspace under Assignment 1, **deadline: 13:30 AM.**

Exercise 1. Let g be a variant of the fib function, defined by $g(0) = 0$, $g(1) = 1$ and $g(n + 2) = g(n + 1) + 2g(n)$ for $n \geq 0$.

(10) (a) Use the method of the lecture to derive a "closed expression" for $g(n)$ (that is: express $g(n)$ in terms of n, without recursion).

(10) (b) Show that g is exponential by giving a such that $g(n) = \Theta(a^n)$. (Argue that your a is correct, a proof using the Substitution Method is not required.)

Exercise 2. An $n \times m$ chocolate bar can be broken into unit squares, by breaking along the unit lines. Claim: the number of breaks required is $nm - 1$.

Prove this by strong induction

Exercise 3. . We prove a number of standard useful properties about log. Suggestion: **remember these**. First of all some facts from which everything basically follows.

- By definition:
$$\log_b a = x \quad \Longleftrightarrow \quad b^x = a$$
- Exponent of log:
$$b^{\log_b a} = a$$
- The function log (and also exponentiation) is injective:
$$\log_b x = \log_b y \quad \Longleftrightarrow \quad x = y.$$
(Proof from left to right: $\log_b x = \log_b y \Rightarrow b^{\log_b x} = b^{\log_b y} \Rightarrow x = y$.)
- The function log (and also exponentiation) is monotone:
$$\log_b x < \log_b y \quad \Longleftrightarrow \quad x < y.$$

Let $a, b, c > 0$. Prove the following.

(a) (Hint: consider a^x for x the two sides of the equation.)
$$\log_a b \cdot \log_b c = \log_a c.$$

(b) From (a), show that one can "change log-base at a constant cost factor", by showing that
$$\log_b f(n) = d \log_a f(n) \text{ for some constant } d > 0.$$

(c) From (a) (hint: take $\log_a$ at both sides of this equation):

$$a^{\log_c b} = b^{\log_c a}.$$

(d) Given $d > 0$, prove that

$$\log_a x + d > \log_a(x + d)$$

for sufficiently large x. When is x "sufficiently large"? (Note that this also implies $\log_a(x - d) > \log_a x - d$. These inequalities are sometimes helpful in approximations to commute log with +.)

Exercise 4. Given $T(n) = 2T(\left\lfloor \frac{2n}{5} \right\rfloor) + T(\left\lfloor \frac{n}{5} \right\rfloor) + d\,n$, where $d \geq 0$. (So d is some fixed constant.)

(20) (a) Prove that $T(n) = \mathcal{O}(n \log n)$, where you take rounding off errors into account.

(b) Prove that $T(n) = \Omega(n \log n)$, where you take rounding off errors into account.

Exercise 5. Let $T(n) = 10T(\frac{n}{3}) + \Theta\left(n^2\right)$. Prove the following using the Substitution Method (where you may ignore rounding off errors).

(a) $T(n) = \mathcal{O}\left(n^2\sqrt{n}\right)$, and

(b) $T(n) = \Omega\left(n^2 \log_3 n\right)$.

Exercise 6. Let $T(n) = 6T(n-2) + 9T(n-3) + \Theta\left(n^2\right)$. Prove the following using the Substitution Method (where you may ignore rounding off errors).

(30) (a) $T(n) = \mathcal{O}(3^n)$. (Hint: you may need to add an additional term $-d\,n^2$.)

(30) (b) $T(n) = \Omega(3^n)$.

Exercise 7. Rank the following functions in n by order of growth from low to high; some may be of the same order. We view f as being "lower than" g in case $f = \mathcal{O}(g)$ and $g \neq \mathcal{O}(f)$.

$$n\sqrt{n} \qquad \sum_{i=0}^{n} \log n \qquad n^n \qquad \log\sqrt{n} \qquad \log(n^2) \qquad (\log n)^2 \qquad 2^n \qquad 3^n$$

$$\sum_{i=0}^{\lfloor \log n \rfloor} i \qquad \sum_{i=0}^{n} i^2 \qquad n^{0.001} \qquad 17n^3 \qquad 17^{\log 89} \qquad n^2 \qquad 100n \qquad 1$$