Exercises Complexity Theory

Lecture 6

May 27, 2024

Only the exercises where points are given can be handed in.
(The maximum number of points per exercise is written in the margin.)

To be handed in on **June 3, 2024**, in Brightspace under Assignment 6, **deadline: 13:30.**

Exercise 1. For a finite set $S \subseteq \mathbb{N}$, we say that subsets $S_1, S_2 \subseteq S$ *partition* S if they are both non-empty, $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$. Now consider the following decision problem Part:

Given a finite set $S \subseteq \mathbb{N}$, does there exist a partition S_1, S_2 of S such that

$$\sum_{s_1 \in S_1} s_1 = \sum_{s_2 \in S_2} s_2?$$

(5) (a) Give two sets (each with at least 3 elements), one which is a *yes*-instance of Part and one which is a *no*-instance.

(15) (b) Show that Part is **NP**-complete. (*Hint* Consider the problem $\mathsf{SubsSum}$.)

Exercise 2. Say for each of the following problems whether they are in **P** or **NP**-hard. Give a proof sketch of your answer.

(10) (a) Given an undirected graph $G = (V, E)$, is there a subset $V' \subseteq V$ such that for each $\{u, v\} \in E$, exactly one of u, v is contained in V'? That is: we are looking for a V' that is a vertex cover and an independent set (no two vertices of V' are connected by an edge).

(10) (b) By an edge contraction on $G = (V, E)$ we mean the operation which yields a graph $G' = (V', E')$ where $V' = (V \setminus \{u, v\}) \cup \{w\}$ for some $\{u, v\} \in E$ with $u \neq v$ and $w \notin V$, and $E' = (E \setminus \{\{u, v\}\}) \cup \{\{e, w\} \mid \{e, u\} \in E \text{ or } \{e, v\} \in E\}$. I.e. we replace an edge and its two endpoints with a single vertex, maintaining edges out of the original pair of vertices. The decision problem of interest: given a graph $G = (V, E)$, can we obtain the complete graph on two nodes K_2 from G by a sequence of edge contractions?

(10) (c) The problem Part from exercise 1 where we now allow finite multisets $S \subseteq \mathbb{N}$ (i.e. each number may occur multiple times in S) and with the additional condition that $|S_1| = |S_2| = \frac{|S|}{2}$. (You may use the results from exercise 1 in your answer).

Exercise 3. A spanning tree for a graph $G = (V, E)$ is a graph $G' = (V, E')$ with $E' \subseteq E$ such that for each pair of nodes $u, v \in V$ there is exactly one path connecting u and v.

The decision problem $\mathsf{BSP}(k)$ is then the problem of deciding for a given graph G whether there is spanning tree for G with each node having degree at most k.

Note: the degree of a node u in a graph $G = (V, E)$ is the number of edges in E with u as an endpoint.

(5) (a) Give a graph G which is a *yes*-instance of $\mathsf{BSP}(2)$ and give a spanning tree G' which witnesses this such that $G \neq G'$.

(10) (b) Is $\mathsf{BSP}(2)$ in **P** or is it **NP**-complete? Prove your answer.

(15) (c) Show that $\mathsf{BSP}(k)$ is **NP**-complete for all $k > 2$.

Exercise 4. For a graph $G = (V, E)$, a *nearly-Hamilton path* is a path in the graph G that has exactly one vertex occurring twice, and all others occurring exactly once. The problem $\mathsf{nearHam}(G)$ is to decide whether G has a nearly-Hamilton path.

(5) (a) Give a graph $G = (V, E)$ that has a nearly-Hamilton path, but not a Hamilton path.

(15) (b) Prove that $\mathsf{nearHam}$ is **NP**-complete.