Complexity IBC028, Lecture 1

H. Geuvers

Institute for Computing and Information Sciences
Radboud University Nijmegen

Version: spring 2024

Outline

About this course I

Lectures

- Teacher: Herman Geuvers
- Weekly, 2 hours, on Monday, 13:30-15:15 (with an exception today)
- The lectures follow:
  - these slides, available via the web
  - extra lecture notes by Hans Zantema, available via the web
  - *Introduction to Algorithms*, "CLRS", by Cormen, Leiserson, Rivest and Stein
    OR *Algorithms Illuminated Omnibus Edition*, "Roughgarden", by Tim Roughgarden.
- Course URL:

  www.cs.ru.nl/~herman/onderwijs/complexity2024/

  Please check there first

About this course II

Exercises

- Weekly exercise classes, on Friday, 10:30-12:15 (or Friday 13:30-15:15, for Double Bachelor Math-CS)
- **First exercise**: register for an exercise class in Brightspace.
- Schedule:
  - Monday: "lecture n" and "exercises n" on the web
  - Next exercise class (Friday) you can work on "exercises n", ask questions, get answers for "exercises $n-1$".
  - Next Monday, before **13:30**: hand in "exercises n" via Brightspace.
  - Before next exercise class: find you grade for "exercises n" in Brightspace
- Your handed in exercises are graded by your TA.
- If e is the average grade of your exercises, $\frac{e}{10}$ is added to your exam grade as a bonus.

About this course IV

Examination

- The final grade is composed of
  - the grade of your final (3hrs) exam, **f**,
  - the average grade of your exercises, **e**,
- The final grade is computed as follows.
  - If **f** is 5 or higher, the final grade is $\min(10, \mathbf{f} + \frac{\mathbf{e}}{10})$
  - If **f** is below 5, the final grade is **f**.
- The re-exam is a full 3hrs exam about the whole course. You keep the (average) grade of the exercises.
- If you fail again, you must start all over next year

Overview

Topics

- Techniques for computing the complexity of algorithms, especially recursive algorithms; substitution method, recursion tree method, the "master theorem".
- Examples of algorithms and data structures and their complexity.
- Complexity classes: P (polynomial complexity), NP; NP-completeness and $P \stackrel{?}{=} NP$?

Important:

$\Longrightarrow$ Precise formal definitions and precise formal proofs

Complexity of algorithms

Time complexity of algorithm $A := \#$ steps it takes to execute A.

- what is a "step"?
- algorithm ... not "program"!
- $\#$ steps should be related to size of input

Time complexity of algorithm A is f if

for an input of size n, A takes $f(n)$ steps to compute the output.

Here, f is a function from $\mathbb{N}$ to $\mathbb{N}$.

- We study worst case complexity: we want an upperboud that applies to all possible inputs.
- We study complexity "in the limit" and ignore a finite number of "outliers": asymptotic complexity
- We ignore constants and lower factors: n^2 and $5n^2 + 3n + 7$ are "the same" complexity.

Asymptotic complexity

Complexity definitions: "big $\mathcal{O}$", "big Ω", "big Θ" notation.
For $f, g : \mathbb{N} \to \mathbb{N}$ a functions,

- $f \in \mathcal{O}(g)$ if $\exists c \in \mathbb{R}_{>0}\, \exists N_0\, \forall n > N_0 (f(n) \leq c\, g(n))$
- $f \in \Omega(g)$ if $\exists c \in \mathbb{R}_{>0}\, \exists N_0\, \forall n > N_0 (c\, g(n) \leq f(n))$
- $f \in \Theta(g)$ if $f \in \mathcal{O}(g) \cap \Omega(g)$.

- $\mathcal{O}(g)$ is a set of functions (and similarly for $\Omega(g)$ and $\Theta(g)$):

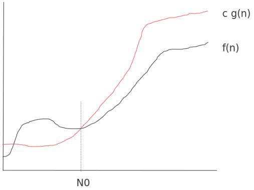
$$\mathcal{O}(g) = \{f \mid \exists c \in \mathbb{R}_{>0}\, \exists N_0\, \forall n > N_0 (f(n) \leq c\, g(n))\}$$

- Nevertheless, one always writes $f = \mathcal{O}(g)$, and we will follow that (abuse of) notation.
- Also: we follow the habit of writing $f(n)$ for the function $n \mapsto f(n)$, so we write $f(n) = \mathcal{O}(g(n))$ etc.

$f(n) = \mathcal{O}(g(n))$

$f(n) = \mathcal{O}(g(n))$ if

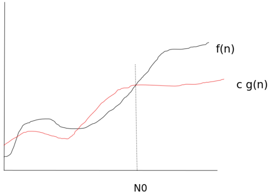
$$\exists c \in \mathbb{R}_{>0}\, \exists N_0\, \forall n > N_0 (f(n) \leq c\, g(n))$$



$f(n) = \Omega(g(n))$

$f(n) = \Omega(g(n))$ if

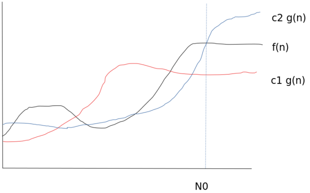
$$\exists c \in \mathbb{R}_{>0}\, \exists N_0\, \forall n > N_0 (c\, g(n) \leq f(n))$$



$f(n) = \Theta(g(n))$

$f(n) = \Theta(g(n))$ if $f(n) = \mathcal{O}(g(n)) \wedge f(n) = \Omega(g(n))$.
This is equivalent to saying:

$$\exists c_1, c_2 \in \mathbb{R}_{>0}\, \exists N_0\, \forall n > N_0 (c_1\, g(n) \leq f(n) \leq c_2\, g(n))$$



Why can we ignore constants and lower factors

For $f(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0$ with $a_k \neq 0$, we have

$$f(n) = \Theta(n^k)$$

We show this by an example: $7n^2 + 5n + 8 = \Theta(n^2)$

Space complexity

Apart from running time as a measure of complexity, one could also look at memory consumption. This is called space

complexity': memory it takes to execute an algorithm. In the final lectures we will say something about space complexity, but for now we restrict to time complexity. Just one observation:

space complexity $\leq$ time complexity, because it takes at least n time steps to use n memory cells.

Strong induction (I)

The induction principle that we have used is also called structural induction: it relies directly on the inductive structure of $\mathbb{N}$.

$$\frac{P(0) \qquad \forall n \in \mathbb{N}\,(P(n) \rightarrow P(n+1))}{\forall n \in \mathbb{N}\,(P(n))}$$

We will often use strong induction, which relies on the fact that $<$ is well-founded on $\mathbb{N}$. (No infinite decreasing $<$-sequences in $\mathbb{N}$.)

Strong induction:

$$\frac{\forall n \in \mathbb{N}\,(\forall k < n(P(k)) \rightarrow P(n))}{\forall n \in \mathbb{N}\,(P(n))}$$

Strong induction gives a stronger induction hypothesis: to prove $P(n)$ we may assume as (IH): $\forall k < n\,(P(k))$ (and not just $P(n-1)$).

Strong induction (II)

Strong induction:

$$\frac{\forall n \in \mathbb{N}\,(\forall k < n(P(k)) \to P(n))}{\forall n \in \mathbb{N}\,(P(n))}$$

Strong induction is only seemingly stronger: in fact the two reasoning principles are equivalent.

Strong induction can be proved by proving $\forall k < n\,(P(k))$ by (structural) induction on n.

Fibonacci (I)

The Fibonacci function is defined as follows.

$$\begin{aligned} \mathrm{fib}(0) &= 0 \qquad \mathrm{fib}(1) = 1 \\ \mathrm{fib}(n+2) &= \mathrm{fib}(n+1) + \mathrm{fib}(n) \end{aligned} \tag{1}$$

Claim: fib is exponential.

- So we are looking for an a such that $\mathrm{fib}(n) = \Theta(a^n)$.
- Let's first try to find an a such that $\mathrm{fib}(n) = a^n$.

Looking at equation (1), a should satisfy

$$a^{n+2} = a^{n+1} + a^n.$$

Knowing that $a \neq 0$, we obtain the quadratic equation $a^2 = a + 1$ that we can easily solve. Its solutions are called φ and $\hat{\varphi}$:

$$\varphi := \frac{1+\sqrt{5}}{2} \approx 1.618 \qquad \hat{\varphi} := \frac{1-\sqrt{5}}{2} \approx -0.618$$

Fibonacci (II)

$$\mathrm{fib}(0) = 0 \qquad \mathrm{fib}(1) = 1$$

$$\mathrm{fib}(n+2) = \mathrm{fib}(n+1) + \mathrm{fib}(n) \tag{1}$$

$$\varphi := \frac{1+\sqrt{5}}{2} \approx 1.618 \qquad \hat{\varphi} := \frac{1-\sqrt{5}}{2} \approx -0.618$$

Neither φ^n nor $\hat{\varphi}^n$ provide solutions to the equations for fib, but

- the sum of two solutions to (1) is again a solution to (1)
- a solution to (1) multiplied with a c is again a solution to (1)

So we try to find c_1 and c_2 such that $\mathrm{fib}(n) = c_1\varphi^n + c_2\hat{\varphi}^n$. This yields a unique solution and we obtain

$$\mathrm{fib}(n) = \frac{1}{5}\sqrt{5}\,\varphi^n - \frac{1}{5}\sqrt{5}\,\hat{\varphi}^n.$$

As $\hat{\varphi}^n \to 0$, we can conclude that $\mathrm{fib}(n) = \Theta(\varphi^n)$.

Binary search trees

A binary search tree, bst, is a binary tree that has, in its nodes and leaves, elements of an ordered structure $(A, \sqsubseteq)$, where for every node labeled a with left subtree ℓ and rightsubtree r,

- for all labels x in ℓ: $x \sqsubseteq a$
- for all labels y in r: $a \sqsubseteq y$.

Often we have $(\mathbb{N}, \leq)$ as ordered structure.

- A bst is an efficient data-structure for storing search data if the tree is balanced: searching in a tree t is efficient if the height t is $\mathcal{O}(\log k)$ for k the number of nodes in t.
- In a previous lecture you have seen red-black trees.
- We now introduce AVL-trees, also because they give a nice application of the fib function.

AVL trees

Definition

An AVL tree is a binary search tree in which, for every node a, the difference between the height of the left and the right subtree of a is ≤ 1.

The following Theorem shows that AVL trees are efficient.

Theorem

The height of an AVL tree t with k nodes is $\mathcal{O}(\log k)$.

The Theorem follows from our result that fib is exponential and a Lemma.

Lemma

The number of nodes in an AVL tree of height n is $\geq \text{fib}(n)$.

The number of nodes in an AVL tree

Lemma

The number of nodes in an AVL tree of height n is $\geq \text{fib}(n)$.

Proof. By (strong) induction on n.

IH: for all $p < n$: if t is an AVL tree of height p, then the number of nodes in t is $\geq \text{fib}(p)$.
To prove: if n is the height of an AVL tree s, then the number of nodes in s is $\geq \text{fib}(n)$.
Case distinction on n:

- $n = 0, 1$. Easy; check for yourself.
- $n \geq 2$. Then $n = 1 + \max(\text{height}(s_1), \text{height}(s_2))$, where s_1 and s_2 are the left and right subtrees of the top node of s. One of s_i has $\text{height}(s_i) = n - 1$, while the other has height $n - 1$ or $n - 2$. Using (IH) we derive that the number of nodes in s is $\geq 1 + \text{fib}(n-1) + \text{fib}(n-2)$, which is $\geq \text{fib}(n)$.

☺

AVL trees are efficient

Theorem

The height of an AVL tree t with k nodes is $\mathcal{O}(\log k)$.

Proof

Let $d(k) :=$ the largest height of an AVL tree with k nodes. So for every k there is an AVL tree with k nodes that has height $d(k)$.
Following the Lemma and our earlier result on fib: there is a $c > 0$ such that: $k \geq c\varphi^{d(k)}$ for all k (larger than some fixed N_0).
Therefore: $\log k \geq \log(c\varphi^{d(k)}) = \log c + d(k) \log \varphi$ and so

$$d(k) \leq \frac{\log k - \log c}{\log \varphi} = \mathcal{O}(\log k)$$

Divide and Conquer algorithms: Mergesort

For A an array p, r numbers, MergeSort(A, p, r) sorts the part $A[p], \dots A[r]$ and leaves the rest of A unchanged.

$$\text{MergeSort}(A,p,r) = \texttt{if } p < r \texttt{ then} \quad \begin{array}{l} q := \left\lfloor \frac{p+r}{2} \right\rfloor; \\ \text{MergeSort}(A,p,q); \\ \text{MergeSort}(A,q+1,r); \\ \text{Merge}(A,p,q,r) \end{array}$$

- Merge(A, p, q, r) merges the parts $A[p], \dots A[q]$ and $A[q+1], \dots A[r]$. It is linear (in the length of A) and produces a sorted array (if the input arrays are sorted).
- We write a recurrence relation for $T(n)$, the time it takes to compute MergeSort(A, p, r), with $n = r - p$

Mergesort

For A an array p, r numbers, MergeSort(A, p, r) sorts the part $A[p], \ldots A[r]$ and leaves the rest of A unchanged.

$$\text{MergeSort}(A, p, r) = \texttt{if } p < r \texttt{ then} \quad \begin{array}{l} q := \left\lfloor \dfrac{p+r}{2} \right\rfloor; \\ \text{MergeSort}(A, p, q); \\ \text{MergeSort}(A, q+1, r); \\ \text{Merge}(A, p, q, r) \end{array}$$

Recurrence equation for T of MergeSort

$$\begin{array}{rcl} T(1) & = & 1 \\ T(n) & = & T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lceil \frac{n}{2} \right\rceil) + \Theta(n) \end{array}$$

How can we solve this and compute T?

The complexity of Mergesort (I)

$$\text{MergeSort}(A,p,r) = \texttt{if } p < r \quad \texttt{then} \quad q := \left\lfloor \frac{p+r}{2} \right\rfloor; \text{MergeSort}(A,p,q); \text{MergeSort}(A,q+1,r); \text{Merge}(A,p,q,r)$$

In computing complexity, we find that we can ignore rounding, so we have

- $T(1) = 1$
- $T(n) = 2T(\frac{n}{2}) + \Theta(n)$ (for $n \geq 2$)

Theorem

If $T(n) \leq 2T(\lfloor \frac{n}{2} \rfloor) + \Theta(n)$, then

$$T(n) = \mathcal{O}(n \log n).$$

The complexity of Mergesort(II)

Theorem

If $T(n) \leq 2T(\lfloor \frac{n}{2} \rfloor) + \Theta(n)$, then $T(n) = \mathcal{O}(n \log n)$.

Proof (by strong induction)

We know that there are $c_0 > 0$ and N_0 such that
$\forall n > N_0 (T(n) \leq 2T(\lfloor \frac{n}{2} \rfloor) + c_0\, n)$.
Need to find: $c_1 > 0$ and N_1 such that $\forall n > N_1 (T(n) \leq c_1\, n \log n)$.
Take $c_1 \geq c_0$ large enough so that $T(n) \leq c_1 n \log n$ for $n = 1, 2, 3$.
Let $N_1 > 3, N_0$. Then for $n > N_1$ we have $\lfloor \frac{n}{2} \rfloor < n$, so we can apply strong induction.

$$
\begin{aligned}
T(n) \leq 2T(\left\lfloor \frac{n}{2} \right\rfloor) + c_0\, n \quad & \overset{\text{IH}}{\leq} \quad 2c_1 \left\lfloor \frac{n}{2} \right\rfloor \log \left\lfloor \frac{n}{2} \right\rfloor + c_0 n \\
& \leq \quad 2c_1 \frac{n}{2} \log \frac{n}{2} + c_1 n \\
& \leq \quad c_1 n(\log n - 1) + c_1 n \\
& = \quad c_1 n \log n \qquad \square
\end{aligned}
$$

Back to Mergesort

For MergeSort, we had $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + \Theta(n)$.
What if, in fact, we "round up" and have

$$T(n) = 2T(\left\lceil \frac{n}{2} \right\rceil) + \Theta(n)?$$

We show that it doesn't matter: If $T(n) \leq 2T(\lfloor \frac{n}{2} \rfloor + D) + c\, n$, for fixed D and c, then $T(n) = \mathcal{O}(n \log n)$.

Define $U(n) := T(n + 2D)$. Then

$$\begin{aligned} U(n) = T(n+2D) &\leq 2T(\left\lfloor \frac{n+2D}{2} \right\rfloor + D) + c(n+2D) \\ &\leq 2U(\left\lfloor \frac{n}{2} \right\rfloor) + 2c\, n \qquad \text{(for } n \geq 2D) \end{aligned}$$

Earlier Theorem: $U(n) = \mathcal{O}(n \log n)$. So we also have $T(n) = \mathcal{O}(n \log n)$. □

Substitution method: Induction loading

$$T(n) = T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lceil \frac{n}{2} \right\rceil) + 1 \qquad \text{for } n \geq 2, \text{ and } T(1) = b$$

We guess that $T(n) = \mathcal{O}(n)$ and we try to show that $T(n) \leq c\, n$ for some appropriately chosen c (and $n > N$ for some chosen N).

$$\begin{array}{rcl} T(n) & \leq & c \left\lfloor \frac{n}{2} \right\rfloor + c \left\lceil \frac{n}{2} \right\rceil + 1 \\ & = & cn + 1 \quad \overset{??}{\leq} cn \quad \ldots \text{ no!} \end{array}$$

The trick is to add some constant: $T(n) \leq c\, n + d$.
Try the proof again and figure out what c and d could be.

$$\begin{array}{rcl} T(n) & \leq & c \left\lfloor \frac{n}{2} \right\rfloor + d + c \left\lceil \frac{n}{2} \right\rceil + d + 1 \\ & = & cn + 2d + 1 \\ & \leq & cn + d \qquad \text{for } d = -1 \text{ and any } c. \end{array}$$

For the base case: $T(1) = b \leq c - 1$, so take $c := b + 1$.
We have $T(n) \leq (b+1)n - 1$ for all $n \geq 1$, so $T(n) \in \mathcal{O}(n)$.

Substitution method: Changing variables

$$T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n$$

We rename variables and put $n = 2^m$ (and so $m = \log n$). Ignoring rounding off errors, we have

$$T(2^m) = 2T(2^{m/2}) + m$$

Consider this as a function in m: $S(m) = T(2^m)$ and we have

$$S(m) = 2S(\frac{m}{2}) + m$$

This is well-known and we have $S(m) = \mathcal{O}(m \log m)$.
We conclude that

$$T(n) = T(2^m) = S(m) \leq c(m \log m) = c(\log n \log \log n)$$

for some c.
So $T(n) = \mathcal{O}(\log n \log \log n)$.

Pitfalls in proving complexity

Suppose $T(1) = 1$ and $T(n) = T(n-1) + n$ for $n > 1$.
Claim: then $T(n) = \mathcal{O}(n)$
Proof: By induction on n:

$$\begin{array}{rcl} T(n) & = & T(n-1) + n \\ & \stackrel{\text{IH}}{=} & \mathcal{O}(n) + \mathcal{O}(n) = \mathcal{O}(n) \end{array}$$

$\Longrightarrow$ This is WRONG! We need to be precise about functions and constants in induction proofs:
$T(n) = \mathcal{O}(n)$ means: $\exists c \, \exists N_0 \, \forall n > N_0 \, (T(n) \leq c\, n)$
Correct reasoning:

$$\begin{array}{rcll} T(n) & = & T(n-1) + n & \\ & \leq & c(n-1) + n & \text{(for } n > N_0\text{)} \\ & = & c\,n + n - c \qquad \not\leq cn & \end{array}$$

and the induction proof doesn't go through.

Substitution method: Example

Given $T(n) = 9T(\frac{n}{2}) + \Theta(n^3)$, prove that $T(n) = \mathcal{O}(n^3\sqrt{n})$.

Some final advice

- Make sure you can do induction proofs. See the exercises.
- Make sure you know how to compute with log, exponents etcetera. That means: you don't have to look up the "rules" but you know them by heart and you can apply them swiftly and correctly. (See e.g. Section 3.2 of the CLRS book.)
- Make sure you know how to compute with summations. (See e.g. Appendix A.1 of the CLRS book.)