Complexity IBC028, Lecture 2

H. Geuvers

Institute for Computing and Information Sciences
Radboud University Nijmegen

Version: spring 2024

Outline

Recursion tree method

The Master Theorem

Techniques to prove $T(n) = \mathcal{O}(g(n))$ [or $T(n) = \Omega(g(n))$ or $T(n) = \Theta(g(n))$]

There are basically three techniques

1. **Substitution Method**:
   Choose g and c (and N_0) and prove (by induction on n)
   $$T(n) \leq c\, g(n) \qquad (\text{for all } n > N_0\ )$$
2. **Recursion Tree method** :
   Method to find g. And then you still have to prove g is correct using (1)
3. **Master theorem method** :
   General theorem for patterns of the shape
   $$T(n) = aT(\frac{n}{b}) + f(n).$$
   Actually: casting the heuristic method of (2) into a general theorem.

Substitution method

Last week (MergeSort):

Theorem

If $T(n) \leq 2T(\lfloor \frac{n}{2} \rfloor) + \Theta(n)$, then

$$T(n) \in \mathcal{O}(n \log n).$$

In fact, the $n \log n$ was an educated guess, which we then proved by induction.

But how do we make an "educated guess"...how do we find the $n \log n$?
Answer: Make a recursion tree!

Recursion Tree method (I)

Example $T(n) = 2T(\frac{n}{2}) + d\,n$.

$$
\begin{array}{ccccccc|c}
 & & & T(n) & & & & d\,n \\
 & T(\frac{n}{2}) & & & & T(\frac{n}{2}) & & 2\,d\,\frac{n}{2} = d\,n \\
T(\frac{n}{4}) & & T(\frac{n}{4}) & & T(\frac{n}{4}) & & T(\frac{n}{4}) & 4\,d\,\frac{n}{4} = d\,n
\end{array}
$$

- The height is $\log n$, so there are $\log n + 1$ layers
- Per layer: $d\,n$ cost contribution
- Bottom: $\#\text{leaves} = 2^{\log n} = n$; cost per leaf $\Theta(1)$.
- Total cost: $d\,n \log n + n\Theta(1)$
- So we conjecture: $T(n) = \Theta(n \log n)$

Some computation rules with log

For exponent: $(b^x)^y = b^{x \cdot y}$ and $b^x\, b^y = b^{x+y}$.

By definition:

$$\boxed{\log_b x = y \iff b^y = x} \qquad \text{and so } b^{\log_b x} = x$$

Rules for log

$$\boxed{\begin{array}{rcl|rcl} \log_b(x \cdot y) & = & \log_b x + \log_b y & \log_b(x^k) & = & k \log_b x \\ \log_b(\frac{x}{y}) & = & \log_b x - \log_b y & \log_b(\frac{1}{x}) & = & -\log_b x \end{array}}$$

Changing base:

$$\boxed{\log_a x = \log_a b \cdot \log_b x} \qquad \text{and so} \qquad \boxed{\log_a f(n) = \log_a b \cdot \log_b f(n)}$$

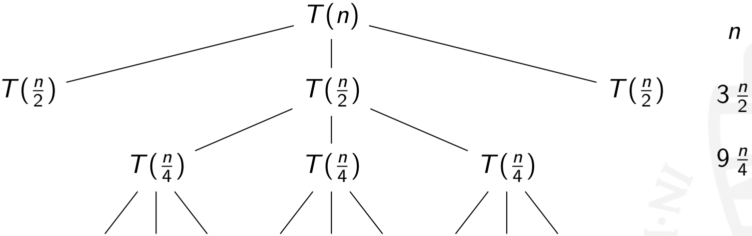
$$\boxed{x^{\log_c y} = y^{\log_c x}} \qquad \text{and so} \qquad \boxed{x^{\log_c f(n)} = f(n)^{\log_c x}}$$

Addition/substraction under log:

$$\boxed{\log(x-1) \geq \log x - 1} \quad \boxed{\log x + 1 \geq \log(x+1)} \quad \text{for } x \geq 2$$

Recursion Tree method (II)

Question. Given $T(n) = 3T(\lfloor \frac{n}{2} \rfloor) + n$, find f with $T(n) = \Theta(f(n))$.



- Height is $\log n$, so $3^{\log n} = n^{\log 3}$ leaves, contributing $\Theta(n^{\log 3})$.
- At layer i we have $3^i \frac{n}{2^i}$ contribution.
- Total: $\Sigma_{i=0}^{\log n} (\frac{3}{2})^i \, n = n \frac{(\frac{3}{2})^{\log n+1} - 1}{\frac{3}{2} - 1} \approx 2n(\frac{3}{2})^{\log n} = 2 \cdot 3^{\log n} = 2 \cdot n^{\log 3}$.
- So we conjecture: $T(n) = \Theta(n^{\log 3})$.

Substitution method

$T(n) = 3T(\lfloor \frac{n}{2} \rfloor) + n$.
We prove: $T(n) = \mathcal{O}(n^{\log 3})$.

Proof. We need to prove $T(n) \leq cn^{\log 3}$ for appropriately chosen c (for all $n > N$ for some appropriately chosen N)

$$\begin{array}{rcl} T(n) & = & 3T(\left\lfloor \frac{n}{2} \right\rfloor) + n \\ & \overset{IH}{\leq} & 3c(\frac{n}{2})^{\log 3} + n \\ & = & \frac{3c\, n^{\log 3}}{2^{\log 3}} + n = cn^{\log 3} + n \overset{??}{\leq} cn^{\log 3} \end{array}$$

The induction fails, so we add a linear factor: $T(n) \leq cn^{\log 3} + dn$.
We notice that it works for $d = -2$, because we have

$$T(n) = 3T(\left\lfloor \frac{n}{2} \right\rfloor) + n \overset{IH}{\leq} 3(c(\frac{n}{2})^{\log 3} - 2\frac{n}{2}) + n = cn^{\log 3} - 3n + n = cn^{\log 3} - 2n$$

Computing the median of an unsorted list

Problem: Given an unsorted list of elements, how to compute the median?
(Median of A = element that has half of the elements of A below it and the other half above it.)
Possible solution:

- First sort the list A, with $|A| = n$.
- Then take the $\left\lfloor \frac{n}{2} \right\rfloor$-th element

This takes $\mathcal{O}(n \log n)$ time.
But it can be done in linear time!

General:

$$M(A, k) := \text{the } k\text{-th element of the sorted version of } A.$$

Then the median of A is $M(A, \frac{|A|}{2})$.

Computing the median of a list in linear time (I)

$$M(A, k) := \text{the } k\text{-th element of the sorted version of } A.$$

Let $n = |A|$. For purpose of exposition, we assume $n = 5^p$ for some p. (If $n < 5^p$ add 0s to get $n = 5^p$.)

1. Split A randomly in $\frac{n}{5}$ groups of 5 elements
2. Determine the median of each group of 5 elements.
3. Determine recursively the median of these $\frac{n}{5}$ medians, say m
4. Count the number of elements in A that are $\leq m$, say ℓ.
   - If $\ell = k$, we are done and m is the output.
   - If $\ell > k$, then m is larger than the number we are looking for, so we continue recursively with $M(A \setminus A_{\text{high}}, k)$
   - If $\ell < k$, then m is smaller than the number we are looking for, so we continue recursively with $M(A \setminus A_{\text{low}}, k - |A_{\text{low}}|)$.
   - Until n is "very small", say $n \leq 10$, then compute the k-th element directly

Q. What exactly are A_{high} and A_{low} and how large are they?

Computing the median of a list in linear time (II)

1. Split A randomly in $\frac{n}{5}$ groups of 5 elements
2. Determine the median of each group of 5 elements.
3. Determine recursively the median of these $\frac{n}{5}$ medians, say m
4. Count the number of elements in A that are $\leq m$, say ℓ.
   - If $\ell = k$, we are done and m is the output.
   - If $\ell > k$, then m is larger than the number we are looking for, so we continue recursively with $M(A \setminus A_{\text{high}}, k)$
   - If $\ell < k$, then m is smaller than the number we are looking for, so we continue recursively with $M(A \setminus A_{\text{low}}, k - 3\left\lceil \frac{n}{10} \right\rceil)$.
   - Until n is "very small", say $n \leq 10$, then compute the k-th element directly

Complexity:

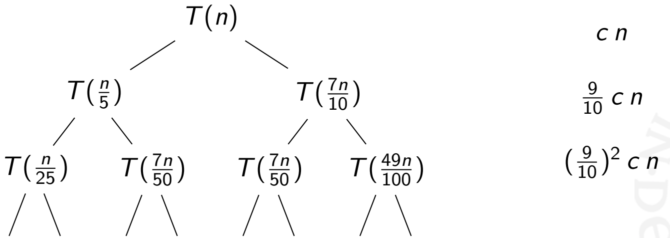
$$T(n) = T(\frac{n}{5}) + T(\frac{7n}{10}) + \Theta(n).$$

Note that steps (1), (2) and the first part of (4) are linear in n.

Computing the median of a list in linear time (III)

$$T(n) \leq T(\frac{n}{5}) + T(\frac{7n}{10}) + cn \text{ for some } c.$$

To find the complexity class of T we can make a recursion tree.



- The height is between $\log_5 n$ and $\log_{\frac{10}{7}} n$, so the number of leaves is approximately $2^{\log_5 n} = n^{\log_5 2}$.
- The layers: $\Sigma_{i=0}^{??}(\frac{9}{10})^i\,c\,n \leq \Sigma_{i=0}^{\infty}(\frac{9}{10})^i\,c\,n = c\,n\,\Sigma_{i=0}^{\infty}(\frac{9}{10})^i = 10\,c\,n$
- Conjecture $T(n) \leq 10\,c\,n$.

Computing the median of a list in linear time (IV)

$$T(n) \leq T(\frac{n}{5}) + T(\frac{7n}{10}) + cn.$$

From the recursion tree method we conjecture that $T(n) \leq 10\, c\, n$.

Proof by induction on n

- For small n, it is correct. (Possibly choose a larger c.)
- For larger n:

$$\begin{array}{rcl} T(n) & \leq & T(\frac{n}{5}) + T(\frac{7n}{10}) + cn \\ & \overset{\text{IH}}{\leq} & 10\, c\, (\frac{n}{5}) + 10\, c\, (\frac{7n}{10}) + c\, n \\ & = & 2\, c\, n + 7\, c\, n + c\, n \\ & = & 10\, c\, n \end{array}$$

So $T(n) = \mathcal{O}(n)$, and so M is linear in the length of the input list.

Master Theorem

Theorem

Suppose $a \geq 1$ and $b > 1$ and we abbreviate $\gamma := \log_b a$.

$$T(n) = aT(\frac{n}{b}) + f(n).$$

Then

1. $T(n) = \Theta(n^\gamma)$ if $f(n) = \mathcal{O}(n^d)$ for some $d < \gamma$.
   f is "relatively small" compared to n^γ
2. $T(n) = \Theta(n^\gamma \log n)$ if $f(n) = \Theta(n^\gamma)$.
   E.g. the Mergesort case
3. $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^d)$ for some $d > \gamma$ and
   $\exists c \in (0,1)\, \exists N\, \forall n > N(a\, f(\frac{n}{b}) \leq c\, f(n))$.
   f is "relatively large" compared to n^γ

Using the Master Theorem (I)

$$T(n) = 9T(\frac{n}{3}) + n$$

Theorem (with $\gamma = \log_b a$)

1. $T(n) = \Theta(n^\gamma)$ if $f(n) = \mathcal{O}(n^d)$ for some $d < \gamma$.
2. $T(n) = \Theta(n^\gamma \log n)$ if $f(n) = \Theta(n^\gamma)$.
3. $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^d)$ for some $d > \gamma$ and $\exists c \in (0,1)\, \exists N\, \forall n > N(a\, f(\frac{n}{b}) \leq c\, f(n))$.

Now, $a = 9$ and $b = 3$, so $\gamma = \log_b a = \log_3 9 = 2$.
Also $f(n) = n = \mathcal{O}(n) = \mathcal{O}(n^1)$ and $1 < 2 = \gamma$.
So case (1) of the Master Theorem applies and we have

$$T(n) = \Theta(n^2).$$

Using the Master Theorem (II)

Theorem (with $\gamma = \log_b a$)

1. $T(n) = \Theta(n^\gamma)$ if $f(n) = \mathcal{O}(n^d)$ for some $d < \gamma$.
2. $T(n) = \Theta(n^\gamma \log n)$ if $f(n) = \Theta(n^\gamma)$.
3. $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^d)$ for some $d > \gamma$ and $\exists c \in (0,1)\, \exists N\, \forall n > N(a\, f(\frac{n}{b}) \leq c\, f(n))$.

$$T(n) = 9\,T(\frac{n}{4}) + n^2.$$

Now, $a = 9$ and $b = 4$, so $\gamma = log_b a = \log_4 9 \approx 1.584$.
Also $f(n) = n^2 = \Omega(n^2)$ and $2 > \gamma$.
So case (3) of the Master Theorem applies and we have

$$T(n) = \Theta(n^2).$$

We need an extra check:
$\exists c \in (0,1)\, \exists N\, \forall n \geq N(a\, f(\frac{n}{b}) \leq c\, f(n))$??
That is: $9(\frac{n}{4})^2 \leq cn^2$, so take $c := \frac{9}{16}$ and this is ok.