Complexity IBC028, Lecture 3

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Outline

Last week: Three techniques to prove complexity

1. **Substitution Method**:
   Choose (guess) f and c (and N_0) and prove $T(n) \leq c\, f(n)$ (for $n > N_0$) by induction on n.
2. **Recursion Tree method** :
   Method to estimate f. And then you still have to prove f is correct using (1)
3. **Master theorem method** :
   General theorem for patterns of the shape

$$T(n) = aT(\frac{n}{b}) + f(n).$$

This week:

- Using the Master Theorem
- Examples of algorithms and applications of the Master Theorem

The Master Theorem

Theorem

Suppose $a \geq 1$ and $b > 1$ and we abbreviate $\gamma := \log_b a$.

$$T(n) = aT(\frac{n}{b}) + f(n).$$

Then

1. $T(n) = \Theta(n^\gamma)$ if $f(n) = \mathcal{O}(n^d)$ for some $d < \gamma$.
   f is "relatively small" compared to n^γ
2. $T(n) = \Theta(n^\gamma \log n)$ if $f(n) = \Theta(n^\gamma)$.
3. $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^d)$ for some $d > \gamma$ and
   $\exists c < 1 \, \exists N \, \forall n > N(a\, f(\frac{n}{b}) \leq c\, f(n))$.
   f is "relatively large" compared to n^γ

The Master Theorem does not always apply (I)

$$T(n) = 2T(\frac{n}{2}) + n \log n.$$

$$T(n) = aT(\frac{n}{b}) + f(n).$$

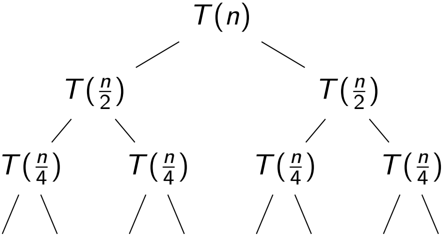
Then, with $\gamma := \log_b a$

1. $T(n) = \Theta(n^\gamma)$ if $f(n) = \mathcal{O}(n^d)$ for some $d < \gamma$.
2. $T(n) = \Theta(n^\gamma \log n)$ if $f(n) = \Theta(n^\gamma)$.
3. $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^d)$ for some $d > \gamma$ and $\exists c < 1 \, \exists N \, \forall n > N (a\, f(\frac{n}{b}) \leq c\, f(n))$.

The Master Theorem does not always apply (II)

$$T(n) = 2T(\frac{n}{2}) + n \log n.$$

We apply the recursion tree method



$$n \log n$$

$$2\,\frac{n}{2} \log \frac{n}{2} = n \log \frac{n}{2}$$

$$4\,\frac{n}{4} \log \frac{n}{4} = n \log \frac{n}{4}$$

So we have

$$T(n) = \Sigma_{i=0}^{\log n} n \log \frac{n}{2^i} + 2^{\log n}\Theta(1) = \Sigma_{i=0}^{\log n} n \log \frac{n}{2^i} + cn$$

Example continued

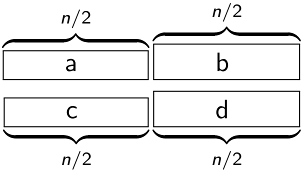
$$T(n) = 2T(\frac{n}{2}) + n \log n.$$

Using the recursion tree method, we have found

$$
\begin{array}{rcl}
T(n) & = & \Sigma_{i=0}^{\log n} n \log \frac{n}{2^i} + cn \\
 & & \text{and by some futher computation:} \\
\Sigma_{i=0}^{\log n} n \log \frac{n}{2^i} & = & n\Sigma_{i=0}^{\log n}(\log n - i) \\
 & = & n(\log^2 n - \Sigma_{i=0}^{\log n} i) \\
 & = & n(\log^2 n - \frac{1}{2}(\log n(\log n + 1)) \\
 & & \text{so we conclude (and this can be proven correct):} \\
T(n) & = & \Theta(n \log^2 n).
\end{array}
$$

Example Karatsuba multiplication (I)

Multiplying two numbers X and Y of n digits in the "standard" way takes $\Theta(n^2)$ steps.
We can do it recursively (assume n is a power of 2):



$$X = a\,2^{n/2} + b$$
$$Y = c\,2^{n/2} + d$$

$$XY = ac\,2^n + (ad + bc)2^{n/2} + bd$$

We do (recursively) 4 multiplications of numbers of size $\frac{n}{2}$, so

$$T(n) = 4T(\frac{n}{2}) + \mathcal{O}(n)$$

So (Master theorem, $a = 4, b = 2$, case (1)): $T(n) = \Theta(n^2)$.
But we can do better.

Example Karatsuba multiplication (II)

Karatsuba multication of two numbers X and Y of n digits.
(Assume n is a power of 2.)

$X =$ [a | b] $= a\,2^{n/2} + b$

$Y =$ [c | d] $= c\,2^{n/2} + d$

Define

$$
\begin{array}{lllllllll}
X_1 & = & a+b & X_2 & = & c+d & X_3 & = & X_1 X_2 \\
X_4 & = & ac & X_5 & = & bd & X_6 & = & X_3 - X_4 - X_5
\end{array}
$$

$X_3 = ac + ad + bc + bd$ and $X_6 = ad + bc$, so XY can be obtained using 3 multiplications of numbers of size $n/2$:

$$T(n) = 3T(\frac{n}{2}) + \mathcal{O}(n)$$

MT, $a = 3, b = 2$, case (1): $T(n) = \Theta(n^{\log_2 3}) \approx \Theta(n^{1.584})$.
NB. In case n is not a power of 2, we add leading zeros.
(Exercise: check that this works and doesn't change the complexity class.)

Matrix multiplication

Multiplying two matrices $A \cdot B$ of size $n \times n$ (with n a power of 2).

- The "standard" algorithm takes $\Theta(n^3)$ steps.
- Any algorithm will be $\Omega(n^2)$.

Recursively: Split A and B into 4 submatrices of order $n/2$:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$\begin{array}{rclcrcl} C_{11} & = & A_{11}B_{11} + A_{12}B_{21} & \quad & C_{12} & = & A_{11}B_{12} + A_{12}B_{22} \\ C_{21} & = & A_{21}B_{11} + A_{22}B_{21} & \quad & C_{22} & = & A_{21}B_{12} + A_{22}B_{22} \end{array}$$

All "administrative steps" are quadratic in n, so we have

$$T(n) = 8\,T(\frac{n}{2}) + \Theta(n^2).$$

Master Theorem ($a = 8$, $b = 2$, case (1)): $T(n) = \Theta(n^3)$.

Strassen Matrix multiplication (I)

Multiplying two matrices can be done in $\Theta(n^{2.8})$ ($\log_2 7 \approx 2.8$).
Split A and B into 4 submatrices of order $n/2$:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$\begin{array}{lllllll} C_{11} & = & A_{11}B_{11} + A_{12}B_{21} & \quad & C_{12} & = & A_{11}B_{12} + A_{12}B_{22} \\ C_{21} & = & A_{21}B_{11} + A_{22}B_{21} & \quad & C_{22} & = & A_{21}B_{12} + A_{22}B_{22} \end{array}$$

We always have to compute each of the C_{ij}. Can we do that with less recursive calls?
Strassen: yes, with 7 calls (in stead of 8)...and a lot of additional administration.

Strassen Matrix multiplication (II)

Define

$$
\begin{array}{lll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} \\
S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} & S_6 = B_{11} + B_{22} \\
S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} \\
S_{10} = B_{11} + B_{12} & &
\end{array}
$$

Define

$$
\begin{array}{lll}
P_1 = A_{11} \cdot S_1 & P_2 = S_2 \cdot B_{22} & P_3 = S_3 \cdot B_{11} \\
P_4 = A_{22} \cdot S_4 & P_5 = S_5 \cdot S_6 & P_6 = S_7 \cdot S_8 \\
P_7 = S_9 \cdot S_{10} & &
\end{array}
$$

Then

$$
\begin{array}{ll}
C_{11} = P_5 + P_4 - P_2 + P_6 & C_{12} = P_1 + P_2 \\
C_{21} = P_3 + P_4 & C_{22} = P_5 + P_1 - P_3 - P_7
\end{array}
$$

Strassen Matrix multiplication (III)

- Strassen's algorithm does 7 recursive calls on a matrix of order $\frac{n}{2}$.
- All "administrative steps" are quadratic in n.

We now have

$$T(n) = 7T(\frac{n}{2}) + \Theta(n^2).$$

Master Theorem ($a = 7$, $b = 2$, $\log_2 7 \approx 2.8074$, case (1)):

$$T(n) = \Theta(n^{\log 7}) \approx \Theta(n^{2.8}).$$

- Later improvements have been made. e.g. to $\Theta(n^{2.3754})$, but Strassen is what is usually implemented.

Matrix inversion (I)

Compute the inverse A^{-1} of a matrix of size $n \times n$. First assume: $A^T = A$. (This is a serious limitation!) So we can write A as

$$\begin{pmatrix} B & C^T \\ C & D \end{pmatrix}$$

Let $S := D - CB^{-1}C^T$. Use standard matrix calculations to show:

$$A^{-1} = \begin{pmatrix} B^{-1} + B^{-1}C^TS^{-1}CB^{-1} & -B^{-1}C^TS^{-1} \\ -S^{-1}CB^{-1} & S^{-1} \end{pmatrix}$$

So we have expressed A^{-1} in terms of B^{-1} and S^{-1} and standard matrix operations. We find that

$$T(n) = 2T(\frac{n}{2}) + \Theta(n^{\log 7})$$

By Master Theorem ($a = b = 2$, $\gamma = \log_2 2 = 1 < \log 7$, case (3)):

$$T(n) = \Theta(n^{\log 7})$$

Check the regularity condition!

Matrix inversion (II)

If A is not symmetric we have (by simple matrix calculations)

$$A^{-1} = (A^T \cdot A)^{-1} \cdot A^T$$

and (!) $A^T \cdot A$ is always symmetric.

Conclusion: A^{-1} can be computed from an inverse of a symmetric matrix + matrix multiplications, so it is in $\Theta(n^{\log 7})$.

In case n is not a power of 2, there is a $k < n$ such that $n + k$ is a power of 2. Apply the algorithm to

$$\begin{pmatrix} A & 0 \\ 0 & I_k \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I_k \end{pmatrix}$$

This becomes max $2\times$ as large, so still $\Theta(n^{\log 7})$.

Matrix calculations

Some remarks

- If matrix multiplication is $\Theta(f(n))$, then taking the inverse is also $\Theta(f(n))$.
- One obtains "inverse" from "multiplication". It can also be done the other way around:
  Given an algorithm for matrix inverse in $\Theta(f(n))$, one can obtain a matrix multiplication algorithm in $\Theta(f(n))$ by

$$\begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix}$$

Minimum distance between points

Given n points in $\mathbb{R}^2$, determine the minimum distance between two points.

- Can be done in $\Theta(n^2)$ steps. (Compute all n^2 distances and determine the minimum.)
- It can also be done in $\Theta(n \log n)$. We present a divide and conquer algorithm MinDist.
- Let a set of points P of size n in $\mathbb{R}^2$ be given.
  We first do some preprocessing:
  - sort P according to the x-coordinate: R_x,
  - sort P according to the y-coordinate: R_y,
  - Cost: $\Theta(n \log n)$.

MinDist algorithm (I)

MinDist(P) =

$$
\begin{array}{lll}
\texttt{if}\ \#P \leq 3 & \texttt{then} & \text{compare the distances } (\leq 3) \\
 & \texttt{else} & P = P_1 \cup P_2, \quad x_\ell, \quad \#P_i \leq \left\lceil \frac{\#P}{2} \right\rceil \\
 & & \forall (x,y) \in P_1 (x \leq x_\ell),\ \forall (x,y) \in P_2 (x \geq x_\ell) \\
 & & \delta_1 := \text{MinDist}(P_1),\ \delta_2 := \text{MinDist}(P_2) \\
 & & \delta := \min(\delta_1, \delta_2) \\
 & & \text{consider } P' := \{(x,y) \mid x_\ell - \delta < x < x_\ell + \delta\} \\
 & & \text{using } R_y \restriction P' \text{ determine, for every } i, \\
 & & \quad \varepsilon_i := \min\{d((x_i,y_i),(x_j,y_j)) \mid i < j \leq i+7\} \\
 & & \delta_3 := \min\{\varepsilon_i \mid 1 \leq i \leq n\} \\
 & & \texttt{return}\ \min(\delta_3, \delta)
\end{array}
$$

MinDist algorithm (II)

- $P = P_1 \cup P_2, \quad x_\ell, \quad \#P_i \leq \left\lceil \frac{\#P}{2} \right\rceil$
- $\forall (x,y) \in P_1 (x \leq x_\ell), \; \forall (x,y) \in P_2 (x \geq x_\ell)$
- $\delta_1 := \mathrm{MinDist}(P_1), \; \delta_2 := \mathrm{MinDist}(P_2), \; \delta := \min(\delta_1, \delta_2)$
- $P' := \{(x,y) \mid x_\ell - \delta < x < x_\ell + \delta\}$

MinDist algorithm (III)

The complexity of MinDist, given a set of points P with $\#P = n$, $T(n)$ is:

$$T(n) = 2T(\frac{n}{2}) + \Theta(n).$$

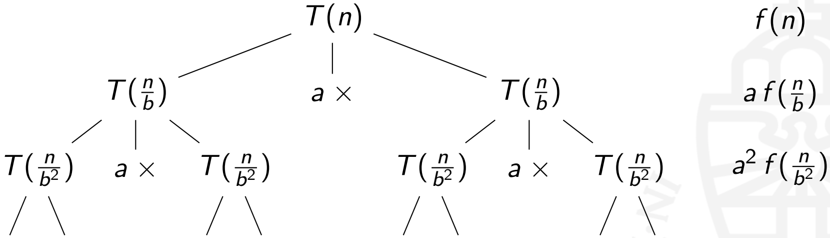
This is a well-known equation (but one can also use the Master Theorem):

$$T(n) = \Theta(n \log n).$$

Together with the preprocessing work (of creating the R_x and R_y orderings), this yields an algorithm of complexity $\Theta(n \log n)$.

Proof sketch of the Master Theorem (I)

Suppose $a \geq 1$ and $b > 1$ and $T(n) = aT(\frac{n}{b}) + f(n)$. Then



- There are $\log_b n$ layers
- There are $a^{\log_b n}$ $(= n^{\log_b a})$ leaves

$$T(n) = \Theta(n^{\log_b a}) + \Sigma_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j}).$$

Proof sketch of the Master Theorem (II)

Let $T(n) = aT(\frac{n}{b}) + f(n)$ (with $a \geq 1, b > 1$) and let $\gamma := \log_b a$.

$$T(n) = \Theta(n^\gamma) + \Sigma_{j=0}^{\log_b n - 1} a^j f(\frac{n}{b^j}).$$

1. If $f(n) = \mathcal{O}(n^d)$ for some $d < \gamma$, then $T(n) = \Theta(n^\gamma)$
2. If $f(n) = \Theta(n^\gamma)$, then $T(n) = \Theta(n^\gamma \log n)$.
3. If $f(n) = \Omega(n^d)$ for some $d > \gamma$,
   and $\exists c < 1 \, \exists N_0 \, \forall n > N_0 \, (a\, f(\frac{n}{b}) \leq c\, f(n)$, then $T(n) = \Theta(f(n))$.