Complexity IBC028, Lecture 6

H. Geuvers

Institute for Computing and Information Sciences
Radboud University Nijmegen

Version: spring 2024

Outline

How to prove that a problem is **NP**-complete?

1. Prove that $A \in$ **NP**: give a polynomial algorithm f such that f verifies A with polynomial certificates, that is:
   $x \in A \Longleftrightarrow \exists y \in \{0,1\}^*(|y| \text{ polynomial in } |x| \wedge f(x,y) = 1)\}$
2. Pick a well-known decision problem B which you know is **NP**-hard,
3. Prove that $B \leq_P A$, that is:
   give a polynomial function h such that

$$x \in B \Longleftrightarrow h(x) \in A.$$

Some **NP**-complete problems (satisfiability)

SAT

- Given a formula φ, is φ satisfiable?
  That is: is there an assignment v such that $v(\varphi) = 1$?

CNF

- Given a formula φ in conjunctive normal form, is φ satisfiable?

$\leq_3$CNF

- Given a formula in "at most 3-conjunctive normal form", is it satisfiable?

3CNF

- Given a formula in "exactly 3-conjunctive normal form", is it satisfiable?

Some **NP**-complete problems (integers)

ILP

- Given an integer linear program, does it have a solution? For example

$$E := \begin{cases} x_1 + 3x_2 - 4x_3 + x_4 & \geq 5 \\ 3x_1 + x_2 + 4x_3 + 2x_4 & \leq 6 \\ 3x_1 - 2x_2 - x_3 - 3x_4 & \geq 0 \end{cases}$$

  Are there $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ such that these inequalities hold?

Some **NP**-complete problems (graphs)

Clique:

- Given a graph $G = (V, E)$ and an integer k, does G have a clique with k vertices?
  That is: is there a set $W \subseteq V$ of size k with an edge between each pair of vertices ?

VertexCover

- Given a graph $G = (V, E)$ and an integer k, does G have a vertex cover with k vertices?
  That is: is there a set $W \subseteq V$ of size k such that each edge "lands in" a vertex in W?

3Color

- Given a graph $G = (V, E)$, does G have a 3-coloring?
  That is: is there a function $c : V \to \{r, y, b\}$ such that $(v, u) \in E \Rightarrow c(v) \neq c(u)$.

How to prove that a problem is **NP**-complete?

- In our proofs of **NP**-hardness we have used the following chain of reductions of satisfiability problems.

$$\text{CNF-SAT} \;\leq_P\; \leq_3\text{CNF-SAT} \;\leq_P\; \text{3CNF-SAT}$$

- We have extended this with proofs of **NP**-hardness of ILP, Clique, VertexCover and 3Color.
- For a proof of **NP**-hardness of Ham-Path (Hamiltonian path), see the note of Niels van der Weide on the webpage, by a reduction: 3CNF-SAT $\leq_P$ Ham-Path.
- In this lecture, we will prove **NP**-hardness of Clique-3Cover, SubsSum (Subset-Sum) WParse (weighted parsing), Ham-Cycle and TSP(traveling salesperson).

A hierarchy of **NP**-completeness proofs

3-Color $\leq_p$ Cliques Cover

CNF-SAT $\leq_p$ $\leq_3$ CNF-SAT $\leq_p$ 3CNF SAT $\leq_p$ Clique $\leq_p$ Vertex Cover

ILP

Subset Sum $\leq_p$ W Parse

HamPath $\leq_p$ HamCycle $\leq_p$ TSP

A hierarchy of **NP**-completeness proofs: last week

see the note on webpage

$\leq_p$ 3-Color $\leq_p$ Cliques Cover

CNF-SAT $\leq_p$ $\leq_3$ CNF-SAT $\leq_p$ 3CNF SAT $\leq_p$ Clique $\leq_p$ Vertex Cover

$\leq_p$ ILP

$\leq_p$ Subset Sum $\leq_p$ W Parse

$\leq_p$ HamPath $\leq_p$ HamCycle $\leq_p$ TSP

A hierarchy of **NP**-completeness proofs: this week

see the note on webpage

$\leq_p$ 3-Color $\leq_p$ Cliques Cover

CNF-SAT $\leq_p$ $\leq_3$ CNF-SAT $\leq_p$ 3CNF SAT $\leq_p$ Clique $\leq_p$ Vertex Cover

$\leq_p$ ILP

$\leq_p$ Subset Sum $\leq_p$ W Parse

$\leq_p$ HamPath $\leq_p$ HamCycle $\leq_p$ TSP

see the note

Clique-3Cover is **NP**-complete

Definition

Clique-3Cover is the problem of deciding if a graph $G = (V, E)$ is the union of three cliques, that is: $\exists V_1, V_2, V_3(V = V_1 \cup V_2 \cup V_3 \wedge V_1 \cap V_2 = \emptyset, V_2 \cap V_3 = \emptyset, V_1 \cap V_3 = \emptyset \wedge \forall i(V_i \text{ is a clique}))$.

Theorem

Clique-3Cover is **NP**-complete

- Clique-3Cover $\in$ **NP**. The sets (V_1, V_2, V_3) are a certificate.
- We show that 3Color $\leq_P$ Clique-3Cover by defining $f(V, E) := (V, \overline{E})$, where $\overline{E} := \{(u, v) \mid u \neq v \wedge (u, v) \notin E\}$.
- (V, E) is 3-colorable iff $(V, \overline{E})$ has a clique-3cover,because

$$\begin{array}{rcl} V_i \text{ is a clique in } (V, \overline{E}) & \Leftrightarrow & \forall u, v \in V_i(u \neq v \rightarrow (u, v) \in \overline{E}) \\ & \Leftrightarrow & \forall u, v \in V_i(u = v \vee (u, v) \notin E) \\ & \Leftrightarrow & V_i \text{ can have one color in } (V, E). \end{array}$$

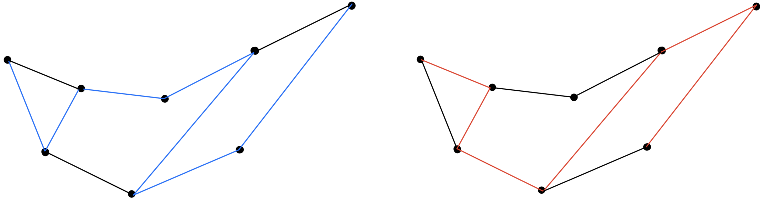
Hamiltonian paths

Definition

Let G be a graph. We say that G has a **Hamiltonian path** if there is a path p in G that crosses every vertex exactly once.

$$\text{Ham-Path} := \{(V,E) \mid \ \exists v_1,\ldots v_n(V = \{v_1,\ldots,v_n\} \ \wedge \ \forall i,j \leq n(v_i = v_j \to i = j) \ \wedge \ \forall i < n(v_i, v_{i+1}) \in E)\}$$

Below, the blue path is Hamiltonian while the red is not.



NP-completeness

We look at the decision problem Ham-Path

- Given a graph G, does G have a Hamiltonian path?

Theorem

Ham-Path *is* **NP***-complete*

It can be shown that 3CNF-SAT $\leq_P$ Ham-Path, See note!

Hamiltonian cycle

Definition

Let G be a graph. We say that G has a **Hamiltonian cycle** if there is a cycle c in G that crosses every vertex exactly once.

$$\text{Ham-Cycle} := \{(V,E) \mid \begin{array}{l} \exists v_1, \ldots v_n (V = \{v_1, \ldots, v_n\} \quad \wedge \\ \forall i,j < n (v_i = v_j \to i = j) \quad \wedge \\ v_n = v_1 \ \wedge \ \forall i < n (v_i, v_{i+1}) \in E)\} \end{array}$$

So, a cycle c is written as $v_1, v_2, \ldots, v_n$ such that $(v_i, v_{i+1}) \in E$ and $(v_n, v_1) \in E$. For a Hamiltonian cycle: $v_i \neq v_j$ if $i \neq j$ and every vertex occurs in this cycle.

Ham-Cycle is **NP**-complete

Theorem

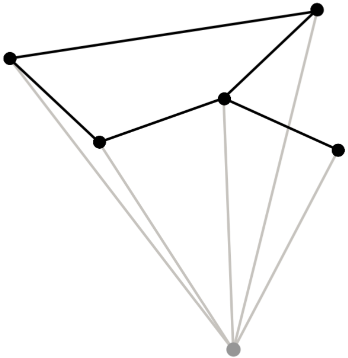
Ham-Cycle *is* **NP***-complete*

Clearly, Ham-Cycle $\in$ **NP**.
To prove that Ham-Cycle is **NP**-hard, we show
Ham-Path $\leq_P$ Ham-Cycle.

- Let $G = (V, E)$ be a graph;
- Add a vertex v and connect it to every vertex $u \in V$ an edge;
- Call the resulting graph G';
- LEMMA
  G has a Hamiltonian path iff G' has a Hamiltonian cycle

Illustration of the proof

From the construction, we get the following graph



Traveling Salesperson, TSP

Definition

Given a **complete** graph G, a cost function on the edges c, and an integer k, is there a cycle in G with cost at most k that crosses every vertex?

$$\text{TSP} := \{(V, c, k) \mid \; c : V \times V \to \mathbb{Z} \wedge k \in \mathbb{Z} \; \wedge \text{there is a cycle with cost at most } k\}$$

Theorem

TSP *is* **NP***-complete.*

Proof

- TSP $\in$ **NP**.
  The certificate is the cycle (the "tour" of the TSP). That it has cost $\leq k$ can be checked easily in polynomial time.

TSP is **NP**-hard

- TSP $\in$ **NPH**. We show Ham-Cycle $\leq_P$ TSP.

  Define for (V, E) a graph the following tuple (V, c, k), consisting of a complete graph, a $c : V \times V \to \mathbb{Z}$, $k \in \mathbb{Z}$.

  - $c(u, v) := 0$ if $(u, v) \in E$,
    $c(u, v) := 1$ if $(u, v) \notin E$
  - $k := 0$

  LEMMA (V, E) has a Hamiltonian cycle if and only if $(V, V \times V)$ has a tour with cost at most 0

  PROOF
  Check $\Rightarrow$ and $\Leftarrow$.

  COROLLARY Ham-Cycle $\leq_P$ TSP and so: TSP is **NP**-hard.

SubsSum, the subset-sum problem

Definition

SubsSum(S, t) is the problem of deciding, for $S \subseteq_{\text{fin}} \mathbb{N}$ and $t \in \mathbb{N}$, if there is a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = t$.
Here, $S \subseteq_{\text{fin}} \mathbb{N}$ denotes that S is a finite subset of $\mathbb{N}$.

Example: take $S = \{1, 4, 6, 9, 12\}$

- There is a subset with sum 14, namely $\{1, 4, 9\}$
- There is no subset with sum 8

We assume the representation of a number $n \in \mathbb{N}$ to be of size $\Theta(\log n)$. This holds for binary or decimal (but for not unary!). For simplicty we now assume decimal representation.

SubsSum is **NP**-complete

Theorem

SubsSum is **NP**-complete

- SubsSum $\in$ **NP**.
  The certificate is the subset $S' \subseteq S$ whose sum is t.
- We prove SubsSum is **NP**-hard by showing $\leq_3$CNF-SAT $\leq_P$ SubsSum.

  We define $f : \leq_3\text{CNF} \to \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathbb{N}$ such that
  $\varphi = \bigwedge_{i=1}^{k} C_i$ is satisfiable if and only if
  for $f(\varphi) = (S, t)$ there is a $S' \subseteq S$ with $\sum_{x \in S'} x = t$.

SubsSum is **NP**-hard

For the definition of $f : \leq_3\text{CNF} \to \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathbb{N}$:

- Assume that $\varphi = \bigwedge_{i=1}^{k} C_i$ has n atoms $\{x_1, \ldots, x_n\}$. S will consist of numbers of $n + k$ digits and t will also have $n + k$ digits.
- Define numbers $p_1, p_1', \ldots, p_n, p_n'$ by:
  - p_i has: 1 at position i and 1 at pos. $n + j$ if x_i occurs in C_j,
  - p_i' has: 1 at position i and 1 at pos. $n + j$ if $\neg x_i$ occurs in C_j,
  - all other positions in p_i and p_i' are 0.
- Define numbers $s_1, s_1', \ldots, s_k, s_k'$ by:
  - s_j has 1 at position $n + j$ and for the rest 0,
  - s_j' has 2 at position $n + j$ and for the rest 0.
- Take $S = \{p_i, p_i' \mid i = 1, \ldots, n\} \cup \{s_j, s_j' \mid j = 1, \ldots, k\}$ and $t = 1 \ldots 14 \ldots 4$ (n times a 1 and k times a 4).
- Lemma: φ is satisfiable iff $\exists S' \subseteq S (\sum_{x \in S'} x = t)$.

$\leq_3$CNF-SAT $\leq_P$ SubsSum: Example

- p_i has 1 at position i and at position $n+j$ if x_i occurs in C_j,
- p'_i has 1 at position i and at position $n+j$ if $\neg x_i$ occurs in C_j.

$$
\begin{array}{lccccc}
(C_1) & x_1 & \vee & \neg x_2 & \vee & \neg x_3 \\
(C_2) & & & \neg x_2 & \vee & x_3 \\
(C_3) & \neg x_1 & \vee & x_2 & & \\
(C_4) & x_1 & \vee & \neg x_2 & \vee & \neg x_3
\end{array}
$$

	n (=3)			k(=4)			
p_1	1	0	0	1	0	0	1
p'_1	1	0	0	0	0	1	0
p_2	0	1	0	0	0	1	0
p'_2	0	1	0	1	1	0	1
p_3	0	0	1	0	1	0	0
p'_3	0	0	1	1	0	0	1

- Basically, the first n colums represent the atoms $x_1, \ldots, x_n$ and the last k colums represent the clauses $C_1, \ldots, C_k$.
- Using a satisfying assignment v for φ, we choose p_i or p'_i for each i (depending on $v(x_i) = 1\,/\,0$).
- Summing up these p's we get $t' = 1\ldots 1 d_1 \ldots d_k$ with $d_j \in \{1,2,3\}$, because ≥ 1 literal in each clause is true.
- So we can add specific s_j and s'_j to sum up to $t = 1\ldots 14\ldots 4$

Parsing and Weighted parsing

- Given a Context Free Grammar (CFG) G and a word w, can we derive **Start** $\Rightarrow w$?
- This is the Parse-problem.
- Put differently: Is there a parse-tree for w?
- The Parse problem can be solved in polynomial time. (E.g. CYK-algorithm)

Variant of the problem WParse, is there a weighted parse tree for w of weight k?

Definition

Given a CFG G where every production rule has a weight, let **Start** $\stackrel{m}{\Rightarrow} w$ denote that w has a parse tree where the sum of the weights of all production rules is m.

WParse(G, w, k) is the problem **Start** $\stackrel{k}{\Rightarrow} w$: Is there a parse tree of w with weight k?

Example: parsing and weighted parsing

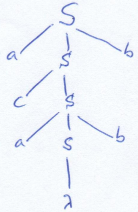
Example

$S \rightarrow aSb$

$S \rightarrow cS$

$S \rightarrow \lambda$

$S \Rightarrow acabb$



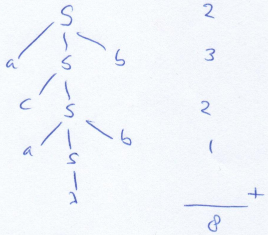
Example

$S \xrightarrow{2} aSb$

$S \xrightarrow{3} cS$

$S \xrightarrow{1} \lambda$

$S \overset{8}{\Rightarrow} acabb$



WParse is **NP**-complete

Theorem

WParse is **NP**-complete

Proof.

1. WParse $\in$ **NP**.
   The certificate is the parse tree of w with weight k
2. We show that WParse is **NP**-hard by showing
   SubsSum $\leq_P$ WParse.
   Given $S = \{s_1, \ldots, s_n\}$ and $k \in \mathbb{N}$ define the following weighted grammar:
   $\mathbf{Start} \xrightarrow{0} A_1 \ldots A_n, \quad A_i \xrightarrow{0} B_i, \quad A_i \xrightarrow{0} \lambda, \quad B_i \xrightarrow{s_i} \lambda.$
   Then

$$\exists S' \subseteq S(\Sigma S' = k) \qquad \text{iff} \qquad \mathbf{Start} \overset{k}{\Rightarrow} \lambda.$$

Decision problems versus optimization problems

VertexCover

- Given a graph G and an integer k, does G have a vertex cover with k vertices?
- Given a graph G, find the **minimal** vertex cover of G.

TSP

- Given a complete graph G, a function c, and an integer k, is there a cycle in G with cost at most k?
- Given a complete graph G and a function c, find a cycle in G with **minimal** cost.

What does **NP**-completeness mean for optimization problems?

- Since we know VertexCover and TSP are **NP**-complete, it will be either difficult or impossible to find efficient algorithms that compute minimal vertex covers or minimal cycles.
- More precisely: If a minimal solution can be computed in polynomial time, then also the decision problem is in **P**.
- But what if we do not go for the **best** solution, but instead for a solution that is **good enough**?

Example: Finding vertex covers (I)

Let $G = (V, E)$ be a graph.
To compute a vertex cover C of G, we take the following steps

- Let $C := \emptyset$ and $E' := E$
- While E' is not empty
  1. Take any edge (u, v) from E'
  2. Take $C := C \cup \{u, v\}$
  3. Remove every edge from E' that touches either u or v.
- output: C

Example: Finding vertex covers (II)

The algorithm is polynomial; it doesn't find the minimal vertex cover...but it is a "decent approximation".

Theorem

The size of the vertex cover computed by the algorithm, is at most twice as large as the size of the minimal vertex cover.

So: while the algorithm does not give the best solution, it gives a solution within reasonable time that may be "good enough" for our purpose.

SAT-solvers

Even though SAT is **NP**-complete, there are very powerful tools that decide whether a (very large) formula is satisfiable. These are called SAT-solvers.

These have been (and are continuously further) optimized and can now deal with tens of thousands of variables and millions of clauses.
SAT-solvers are the "automation workhorses" of computer science.

Example: negative solution to the "Boolean Pythagorean triples problem".

See: Master Course Automated Theorem Proving.

Harder than **NP**

- There are problems that don't have a polynomial checking algorithm, or for which the certificate is not polynomial.
- Example: Two-player games.
  - "Is there a winning strategy for player 1?"
  - Certificate is typically not polynomial size.

Next natural level after **P** (and **NP**): decision algorithms that are polynomially bound on **space** (memory use), not on time.

Definition

f is a polynomial space algorithm for A if

- f is a deterministic Turing Machine that
- halts on every input w such that
- $w \in A$ iff $f(w)$ halts in a final state and
- the size of the tape used in the computation of $f(w)$ is polynomial in $|w|$.

PSPACE

$$\mathbf{PSPACE} := \{A \subseteq \{0,1\}^* \mid \exists f, f \text{ polynomial space algorithm}, w \in A \iff f(w) = 1\}$$

Lemma

- **P** $\subseteq$ **PSPACE**
  Because in polynomial size time, f uses only polynomial size space.
- **NP** $\subseteq$ **PSPACE**
  Because if $A = \{w \mid \exists y(y < c|w|^k \wedge f(w,y) = 1\}$, this can be checked using polynomial size space, by summing up all (exponentially many!) candidate y's and running $f(w,y)$.

NPSPACE

Just like **NP**, we also have **NPSPACE**.

Definition

f is a non-deterministic polynomial space algorithm for A if

- f is a non-deterministic Turing Machine that
- halts on every input w such that
- $w \in A$ iff $f(w)$ has a computation that halts in a final state,
- the size of the tape used in the computation of $f(w)$ is polynomial in $|w|$.

Savitch' Theorem

$$\mathbf{PSPACE} = \mathbf{NPSPACE}$$

PSPACE-hard and PSPACE-complete

Definition

- A is called **PSPACE**-hard if

$$\forall A' \in \textbf{PSPACE}(A' \leq_P A).$$

  (All **PSPACE**-problems can be poly. time reduced to A.)
- **PSpaceH** $:= \{A \mid A$ is **PSPACE**-hard$\}$.
- A is called **PSPACE**-complete if $A \in$ **PSPACE** and A is **PSPACE**-hard.
- **PSpaceC** $:=$ **PSPACE** $\cap$ **PSpaceH**.

Theorem

If $A' \leq_P A$ and $A' \in$ **PSpaceH**, then $A \in$ **PSpaceH**.

The proof is the same as for **NP**-hard.

How to prove that A is **PSPACE**-complete?

Just like SAT is the canonical **NP**-hard problem, there is a canonical **PSPACE**-hard problem: QBF.

Definition

A **quantified boolean formula** (QBF) is a boolean formula where we can now also use quantifiers ($\forall$, $\exists$) over boolean variables.

QBF is the problem of deciding whether a closed quantified boolean formula φ is true.

QBF is **PSPACE**-complete

Example $\varphi = \forall x\, (\exists y\, (x \wedge y)) \vee (\exists z(\neg x \wedge \neg z))$

- For $x = 1$ we can choose $y = 1$ and for $x = 0$ we can choose $z = 0$.
- That is: for all values of x we can choose a case and a value for y (or z) that makes the boolean formula true.
- So φ is true.

Theorem

QBF is **PSPACE**-complete.

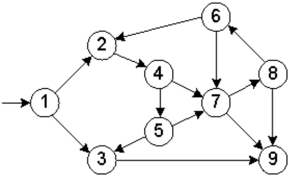
- The "certificate" for QBF(φ) is not just a choice of $0\,/\,1$ for every $\exists$, but a choice depending on the $\forall$ in front of the $\exists$.
- The proof that QBF is **PSPACE**-hard uses a translation of Turing Machines to QBF.

Some variations on QBF

- Note that SAT $\leq_P$ QBF:
  given φ add $\exists x$ in front of φ for all atoms x in φ.
- If we limit QBF to prenex fomulas, that have all quantifiers in front, it is still **PSPACE**-complete.
- If we limit QBF to alternating prenex fomulas, that have alternating $\forall/\exists$ in front, it is still **PSPACE**-complete.
- A "proof" of $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n(\varphi)$ amounts to making n choices, which amounts to a "certificate" of size 2^n.
- A formula like $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n(\varphi)$ can be interpreted as the question for a winning strategy for a two-player game.

Some other **PSPACE**-complete problems

- Strategic games are typically **PSPACE**-complete, like Geography



- Also RushHouR and Sokoban are **PSPACE**-complete.
- Given two regular expression e_1 and e_2, do we have $\mathcal{L}(e_1) = \mathcal{L}(e_2)$? This problem is **PSPACE**-complete. Similarly: Equivalence problem for non-deterministic finite automata: Given two NFAs over Σ, do they accept the same language? (Note: for DFAs this problem is in **P**!)
- The word problem for deterministic context-sensitive grammars is **PSPACE**-complete. This is the problem whether **Start** $\Rightarrow w$ in such a grammar.