# PwCA 2IMF15 Exercises on Normalization

## Some answers

(1) Recall that $\sigma_1 \to \sigma_2 \cdots \sigma_n \to \alpha$ should be read as
$$\sigma_1 \to (\sigma_2 \to ( \cdots \to (\sigma_n \to \alpha) \cdots ))$$
and this means that every type $\sigma \to \tau$ can be written as
$$\sigma_1 \to \tau_1 \to \cdots \to \tau_n \to \alpha$$
where $\tau = \tau_1 \to \cdots \to \alpha$
by decomposing $\tau$ as far as possible.

We have two definitions of $h$:

$$h_1(\alpha) = 0 \qquad h_1(\sigma_1 \to \cdots \to \sigma_n \to \alpha) = \max(h_1(\sigma_1) \ldots h_1(\sigma_n)) + 1$$
$$h_2(\alpha) = 0 \qquad h_2(\sigma \to \tau) = \max(h_2(\sigma)+1, h_2(\tau))$$

We want to show $h_1(\sigma) = h_2(\sigma)$ $(\forall \sigma \in \text{type})$
which we do by induction on $\sigma$

- Case $\sigma = \alpha$ ✓
- case $\sigma = \tau_1 \to \cdots \to \tau_n \to \alpha$

$$\begin{aligned}
h_2(\sigma) &= \max(h_2(\tau_1)+1, h_2(\tau_2 \to \cdots \to \tau_n \to \alpha)) \\
&\overset{IH}{=} \max(h_1(\tau_1)+1, h_1(\tau_2 \to \cdots \to \tau_n \to \alpha)) \\
&= \max(h_1(\tau_1)+1, \max(h_1(\tau_1), \ldots h_1(\tau_n)) + 1) \\
&= \max(h_1(\tau_1), \ldots h_1(\tau_n)) + 1 \\
&= h_1(\sigma)
\end{aligned}$$

$\boxtimes$

PwCA 2IMF 15

Ex. on Normalitätable

(2) In $M$ there are two redexes:

* $II$, whose height is the height of the type of the $I$ on the left.
  Its type is $C$, so $h(II) = h(C)$
* the whole term $M$, whose height is the height of the type of $\lambda x{:}A.\,x(\lambda v{:}B.\,x\,I)$,
  which is $A \to B$, so the height is $h(A \to B)$

This means that the redex with greatest height is the whole term $M$, so we contract that redex.

NB
$h(B) = h(\alpha \to \alpha) = h(\alpha) + 1 = 1$
$h(C) = h(B) + 1 = 2$
$h(A) = \max(h(C)+1, h(B)) = h(C)+1 = 3$

For $m(M)$ we look at height of the redex of maximum height.
this is $h(A \to B) = h(A) + 1 = 4$.
So $m(M) = (4, 1)$

In $N$ we have 4 redexes, 2 of height $h(C)$, the $II$ ones, and 2 of height $h(C \to B)$, the ones with $\lambda z{:}C.\,z(II)$ as function

So we have 2 redexes of max height $h(C \to B) = 3$
So $m(N) = (3, 2)$

So $m(M) > m(N)$.