Proving with Computer Assistance, 2IMF15

Herman Geuvers, TUE

Exercises on Simple Type Theory à la Curry: assigning types to untyped terms, principal type algorithm, some answers

On simple type theory à la Church

Find the answers to these on the answer sheet of last week.

1. Find a term of type $(\delta{\to}\delta{\to}\alpha){\to}(\alpha{\to}\beta{\to}\gamma){\to}(\delta{\to}\beta){\to}\delta{\to}\gamma$
2. Find two terms of type $(\delta{\to}\delta{\to}\alpha){\to}(\gamma{\to}\alpha){\to}(\alpha{\to}\beta){\to}\delta{\to}\gamma{\to}\beta$
3. Find a term of type $((\alpha{\to}\beta){\to}\alpha){\to}(\alpha{\to}\alpha{\to}\beta){\to}\alpha$
4. Find a term of type $((\alpha{\to}\beta){\to}\alpha){\to}(\alpha{\to}\alpha{\to}\beta){\to}\beta$ (Hint: use the previous exercise.)

On simple type theory à la Curry

1. Determine the most general unifiers of

   (a) $(\alpha{\to}\beta){\to}\gamma$ and $\alpha{\to}\beta{\to}\gamma$

   (b) $(\alpha{\to}\beta){\to}\gamma$ and $\gamma{\to}\alpha{\to}\beta$

   ANSWER: For clarity I first insert the brackets:

   (a) $(\alpha{\to}\beta){\to}\gamma \simeq \alpha{\to}(\beta{\to}\gamma)$, yielding $\alpha{\to}\beta \simeq \alpha$ and $\gamma \simeq \beta{\to}\gamma$. The first equation can't be fulfilled, so : "false".

   (b) $(\alpha{\to}\beta){\to}\gamma \simeq \gamma{\to}(\alpha{\to}\beta)$, yielding $\alpha{\to}\beta \simeq \gamma$ and $\gamma \simeq \alpha{\to}\beta$, yielding as a solution $\gamma := \alpha{\to}\beta$. This substitution is the most general unifier.

2. Compute the principal type of $\mathbf{S} := \lambda x.\lambda y.\lambda z.x\,z(y\,z)$.

   ANSWER: We assign type variables to all abstracted variables and applicative subterms:

   $\lambda x{:}\alpha.\lambda y{:}\beta.\lambda z{:}\gamma.((x\,z)^{\delta}(y\,z)^{\varepsilon})^{\zeta}$.

   The type of this term is now $\alpha{\to}\beta{\to}\gamma{\to}\zeta$.

   To make this term typable, we have to solve the following equations.

$$\begin{array}{rcl} \alpha & \simeq & \gamma{\to}\delta \\ \beta & \simeq & \gamma{\to}\varepsilon \\ \delta & \simeq & \varepsilon{\to}\zeta. \end{array}$$

   Substituting the third gives

$$\begin{array}{rcl} \alpha & \simeq & \gamma{\to}\varepsilon{\to}\zeta \\ \beta & \simeq & \gamma{\to}\varepsilon \\ \delta & \simeq & \varepsilon{\to}\zeta. \end{array}$$

   This has the shape of a substitution (only variables to the right; none of the variables at the right occurs at the left), so we have found the most general unifier.

   The principal type is now (substituting in the type $\alpha{\to}\beta{\to}\gamma{\to}\zeta$ above):

$$(\gamma{\to}\varepsilon{\to}\zeta){\to}(\gamma{\to}\varepsilon){\to}\gamma{\to}\zeta$$

3. Which of the following terms is typable? If it is, determine the *principal type*; if it isn't, show that the typing algorithm fails.

   (a) $\lambda z\, x.z(x(\lambda y.y\, x))$
   ANSWER: We assign type variables to all abstracted variables and applicative subterms:
   $$\lambda z{:}\alpha.\lambda x{:}\beta.(z(x(\lambda y{:}\gamma.(y\, x)^{\delta}))^{\varepsilon})^{\zeta}$$
   The type of this term is now $\alpha{\to}\beta{\to}\zeta$.
   To make this term typable, we have to solve the following equations.
   $$\begin{array}{lcl} \gamma & \simeq & \beta{\to}\delta \\ \beta & \simeq & (\gamma{\to}\delta){\to}\varepsilon \\ \alpha & \simeq & \varepsilon{\to}\zeta \end{array}$$
   Substituting the second, we get
   $$\begin{array}{lcl} \gamma & \simeq & ((\gamma{\to}\delta){\to}\varepsilon){\to}\delta \\ \beta & \simeq & (\gamma{\to}\delta){\to}\varepsilon \\ \alpha & \simeq & \varepsilon{\to}\zeta \end{array}$$
   The first equation is not solvable, so: "false".

   (b) $\lambda z\, x.z(x(\lambda y.y\, z))$
   ANSWER: We assign type variables to all abstracted variables and applicative subterms:
   $$\lambda z{:}\alpha.\lambda x{:}\beta.(z(x(\lambda y{:}\gamma.(y\, z)^{\delta}))^{\varepsilon})^{\zeta}$$
   The type of this term is now $\alpha{\to}\beta{\to}\zeta$.
   To make this term typable, we have to solve the following equations.
   $$\begin{array}{lcl} \gamma & \simeq & \alpha{\to}\delta \\ \beta & \simeq & (\gamma{\to}\delta){\to}\varepsilon \\ \alpha & \simeq & \varepsilon{\to}\zeta \end{array}$$
   Substituting the first, we get
   $$\begin{array}{lcl} \gamma & \simeq & \alpha{\to}\delta \\ \beta & \simeq & ((\alpha{\to}\delta){\to}\delta){\to}\varepsilon \\ \alpha & \simeq & \varepsilon{\to}\zeta \end{array}$$
   Substituting the last, we get
   $$\begin{array}{lcl} \gamma & \simeq & (\varepsilon{\to}\zeta){\to}\delta \\ \beta & \simeq & (((\varepsilon{\to}\zeta){\to}\delta){\to}\delta){\to}\varepsilon \\ \alpha & \simeq & \varepsilon{\to}\zeta \end{array}$$
   and we have our most general unifier. So, the principal type is
   $$(\varepsilon{\to}\zeta){\to}((((\varepsilon{\to}\zeta){\to}\delta){\to}\delta){\to}\varepsilon){\to}\zeta.$$

4. Consider the following two terms

   - $(\lambda x.\lambda y.x(\lambda z.y))\,(\lambda w.w)$
   - $(\lambda x.\lambda y.y(\lambda z.y))\,(\lambda w.w)$

For each of these terms, compute its principal, if it exists. (Give the end result and show your computation; if the term has no principle type, show how your computation yields 'fail'.)

ANSWER (partial): the first is typable, the second is not.

5. Compute the principal type of $M := \lambda x.\lambda y.x(y(\lambda z.x\,z\,z))(y(\lambda z.x\,z\,z))$.

6. For each of the following two terms, compute its principal type, if it exists.

   - $\lambda x.(\lambda y.x\,(x\,y))\,(\lambda u\,v.u)$
   - $\lambda y.(\lambda x.x\,(x\,y))\,(\lambda u\,v.u)$

   Give the end result and show your computation; if the term has no principal type, show how your computation yields 'fail'.

   ANSWER (partial): The first one is typable, the second is not. We show the computation of the principal type for the first term. We annotate the abstracted variables and the applicative subterms with type variables:

$$\lambda x^A.((\lambda y^B.(x\,(x\,y)^E)^F)\,(\lambda u^C\,v^D.u))G \qquad : \qquad A \to G$$

   We generate the set of equations that we need to solve:

$$\left\{\begin{array}{rcl} A & = & B \to E \\ A & = & E \to F \\ B \to F & = & (C \to D \to C) \to G \end{array}\right.$$

   We solve it:

$$\left\{\begin{array}{rcl} A & = & B \to E \\ A & = & E \to F \\ B \to F & = & (C \to D \to C) \to G \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} A & = & B \to E \\ A & = & E \to F \\ B & = & C \to D \to C \\ F & = & G \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{rcl} A & = & (C \to D \to C) \to E \\ A & = & E \to G \\ B & = & C \to D \to C \\ F & = & G \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} A & = & E \to G \\ E \to G & = & (C \to D \to C) \to E \\ B & = & C \to D \to C \\ F & = & G \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{rcl} A & = & E \to G \\ E & = & C \to D \to C \\ G & = & E \\ B & = & C \to D \to C \\ F & = & G \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} A & = & (C \to D \to C) \to G \\ E & = & C \to D \to C \\ G & = & C \to D \to C \\ B & = & C \to D \to C \\ F & = & G \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{rcl} A & = & (C \to D \to C) \to C \to D \to C \\ E & = & C \to D \to C \\ G & = & C \to D \to C \\ B & = & C \to D \to C \\ F & = & C \to D \to C \end{array}\right.$$

   This is a substitution, so we are done. Applying this substitution to the type $A \to G$, we get the principal type, which is:

$$((C \to D \to C) \to C \to D \to C) \to C \to D \to C.$$